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ON THE APPLICATION OF SPECTRAL BANDS IN DETERMINING ANOMALOUS DISPERSION AND ACHROMATIC SYSTEMS OF VARIOUS TYPES.

By D. B. Brace.

If two systems of interfering rays, after undergoing different phase-changes, in whatever way, are examined spectrally, dark bands are found for those frequencies where destructive interference occurs. If two sets be examined in juxtaposition, they may be made to coincide, either throughout the entire range of the spectrum, or only over certain portions. The former indicates perfect, the latter partial, differential achromatism. In general, the first is unrealizable; but the latter may be attained to a greater or less extent, particularly in the visible spectrum, by means of certain elements which constitute, in special combinations, achromatic systems.

If N represents the number of times destructive interference, by whatever mode, occurs for such a system, in varying the wave-length from $\lambda = \lambda_\infty$ to $\lambda = \lambda_D$, say, then $[\partial N/\partial\lambda]\partial\lambda$ represents the number of bands in the spectrum between wave-lengths λ and $\lambda + \partial\lambda$. If, by any change, we can make this number the same for a second element, we should have

$$\frac{\partial N'}{\partial \lambda} = \frac{\partial N''}{\partial \lambda}. \qquad (1)$$

The pair is then said to be differentially achromatic for these two wave-lengths. If the bands coincide between these limits, this

couple is perfectly achromatic within this region. In the former, residual spectral effects occur which produce chromatism to a greater or less degree, which latter also occurs outside these limits.

If we integrate (1), we have

$$N' = N'' + k. \tag{2}$$

This shows that if we should vary λ in each element from λ_∞ to λ_D say, where achromatism occurs, and count the number of times destructive interference occurs; or if we should begin in the spectrum at λ_∞ and count the bands up to λ_D, we should, in general, find a different number or order for each element of the pair.

N may be readily obtained if we can vary it in any way as I have already shown.[1] Thus if N_∞, N_D, and N_x represent the orders for λ_∞, λ_D, and λ_x respectively, and N_∞', N_D', and N_x' the orders obtained after any variation, we shall have:

$$N_D - N_\infty = N_D \qquad N_D' - N_\infty' = N_D'$$
$$N_x - N_\infty = N_x \qquad N_x' - N_\infty' = N_x'$$

or

$$N_x - N_D = p$$

and

$$N'_x - N_D' = np,$$

if $N_\infty' = N_\infty$ and p is the number of bands between the two wavelengths λ_D and λ_x, and np the number when N is changed to N'.

We also have

$$N_D' - N_D = m$$

and

$$\frac{N_x' - N_D'}{N_x - N_D} = n = \frac{N_D'}{N_D} = \frac{N_x'}{N_x},$$

since

$$\frac{N_D' - N_\infty'}{N_D - N_\infty} = \frac{N_D'}{N_D},$$

if

$$N_\infty = N_\infty' = 0$$

where m is the number of bands passing λ_D in the positive direction, toward the red, say, in changing N_D to N_D'.

[1] Phil. Mag., Oct., 1899, p. 345; also Rendtorff, Phil. Mag., May, 1901, p. 539, and Williams, PHYS. REV., April, 1904, p. 280.

Hence,

$$N_D = \frac{m}{n-1}. \tag{3}$$

If for λ_x, $N_x \neq 0$, or $= \varphi$, say, then (3) holds only for the corresponding portion or phase φ of the bands. Thus if $\varphi = \lambda/2$, we must use the white bands instead of the black bands in the settings and the countings to obtain N. The true order will then be $N+\varphi$. We have thus the methods at hand of determining the degree of achromatism and also the orders which must be used to reduce the chromatism to a minimum. A similar method will give the order of the same space without the medium, or we may find the thickness by determining the number of air-bands N_{E-D} between two wave-lengths λ_D and λ_E, say. We shall then have at once the data from which we can find the thickness. Thus, if

$$\mu_0 = 1, \quad \frac{d}{\lambda_E} - \frac{d}{\lambda_D} = N_{E-D} \text{ or } d = N_{E-D}\left(\frac{\lambda_D \lambda_E}{\lambda_D - \lambda_E}\right).$$

We have thus the absolute index needed in the technical applications of these principles, which will be embodied in the usual formulæ after transformation from the above equations.

Anomalous Dispersion.

In this case we have to deal with relative phase-retardations of the two interfering systems due to a difference in path, neglecting for the moment phase-changes at the bounding surface.

If d is the thickness and μ the index of the medium,

$$N = \frac{d\mu}{\lambda}; \text{ and } N_0 = \frac{d_0\mu_0}{\lambda} \tag{4}$$

for the same space with a medium of constant index μ_0.

Thus

$$\frac{\partial N}{-\partial \lambda} = \frac{d}{\lambda^2}\left(\mu + \lambda \frac{\partial \mu}{-\partial \lambda}\right) \text{ and } \frac{\partial N_0}{-\partial \lambda} = \frac{d_0\mu_0}{\lambda^2}. \tag{5}$$

If we should be able to adjust the two sets of bands to coincidence at any point of the spectrum, we should have from (1)

$$\frac{\partial N}{-\partial \lambda} = \frac{\partial N_0}{-\partial \lambda}.$$

Hence,

$$\frac{d\mu}{\lambda} + \frac{\partial(d\mu)}{-\partial\lambda} = \frac{d_0\mu_0}{\lambda} \text{ or } \frac{d\mu}{\lambda} - \frac{d_0\mu_0}{\lambda} = -\frac{\partial(d\mu)}{-\partial\lambda} = k = N - N_0. \quad (6)$$

Since, if there is a coincidence between the bands with constant index and those of the medium to be examined, k must be a whole number, we shall have

$$\frac{\partial(d\mu)}{-\partial\lambda} = 0, \quad (7)$$

if $N = N_0$. Thus, if by trial we can obtain a set of coincidences and then find, by the method of counting given above, that the orders are the same, μ must be constant in this region and hence be a maximum or a minimum, or have a point of inflection ; that is, the tangent to the curve of dispersion is parallel to the coördinate axis of λ. For example, taking $\mu_0 = 1$, as in the case of a vacuum

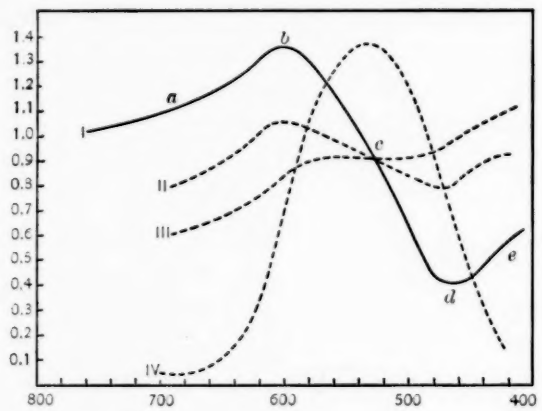

Fig. 1.

(or of air, approximately), let us compare the air-bands with those of fuchsin, say. In the curve of dispersion of the latter, Fig. 1, as determined by Cartmel,[1] equation (7) holds at b and d on curve I. and the order N_1 of the air bands is the same as N_2 that of fuchsin at b ; and N_1' and N_2' are the same at d. Curve II. may illustrate a reduction in the anomaly, in a solution, say, and Curve III. a still further reduction, the points of maximum and minimum coinciding

[1] Phil. Mag., Aug., 1903, p. 213.

at a point of inflection c. At each of these points a characteristic distribution of the bands occurs from which we can readily interpret the nature of the anomaly. Referring to (5) again, since at these points

$$\frac{d\mu}{\lambda^2} = \frac{d_0\mu_0}{\lambda^2},$$

we have the residual term $d/\lambda\, \partial\mu/ - \partial\lambda$ from which to determine the character of the curve at these points.

Thus $\partial\mu/ - \partial\lambda$ is + going from a to b and − going from b to c, or

$$\frac{\partial N}{-\partial\lambda} > \frac{\partial N_0}{-\partial\lambda}$$

and

$$\frac{\partial N}{-\partial\lambda} < \frac{\partial N_0}{-\partial\lambda},$$

respectively; *i. e.*, on the red side of a coincidence, the bands crowd together; while, on the violet side, they spread out relatively to the air-bands in case of a maximum. Going from c to d $\partial\mu/ - \partial\lambda$ is − and from d to e it is +, or

$$\frac{\partial N}{-\partial\lambda} < \frac{\partial N_0}{-\partial\lambda}$$

and

$$\frac{\partial N}{-\partial\lambda} > \frac{\partial N_0}{-\partial\lambda},$$

respectively, *i. e.*, if, on the red side of a coincidence, the bands of the medium spread out and on the violet side they crowd together more than the air-bands of the same order, the region has a minimum index. On Curve III., at a point of inflexion c, $\partial\mu/ - \partial\lambda$ is + on the red and + on the violet side, or

$$\frac{\partial N}{-\partial\lambda} > \frac{\partial N_0}{-\partial\lambda}$$

in each case; *i. e.*, if the bands shorten with respect to the air bands of the same order on each side of a coincidence, the index has a point of inflexion.

In the region beyond a, since the bands must begin to coincide again before $\lambda = \lambda_\infty$, they must begin to lengthen relatively to the

air-bands so as to coincide again at $\lambda = \lambda_\infty$. These various characteristics are illustrated in Fig. 2.

This apparent crowding together and spreading out of the bands is the more marked, the more pronounced the anomaly, being more marked in band- and line-spectra.

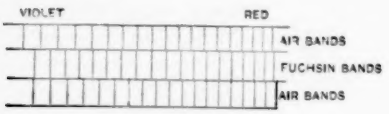

Fig. 2.

If the index of the medium of comparison is not constant, as it is in the case of a vacuum (or air) but varies slowly and uniformly, we can still apply similar tests for any anomalies which may exist. Thus from

$$\frac{\partial N}{-\partial \lambda} = \frac{\partial N_0}{-\partial \lambda}$$

we should have, in such a case,

$$\frac{d\mu}{\lambda} + \frac{\partial (d\mu)}{-\partial \lambda} = \frac{d_0\mu_0}{\lambda} + \frac{\partial (d_0\mu_0)}{-\partial \lambda}$$

or

$$\frac{\partial (d\mu)}{-\partial \lambda} = \frac{(d_0\mu_0)}{-\partial \lambda}, \text{ if } N = N_0,$$

and

$$\frac{\partial \mu}{-\partial \lambda} = \frac{d_0}{d} \frac{\partial \mu_0}{-\partial \lambda}. \tag{8}$$

Since d_0/d will not be large (rarely, if ever, equal to 2 for liquids and solids) and as the inclination of the dispersion curve to the λ-axis is always small for frequencies which are not near the natural frequencies of the medium, $\partial\mu/\partial\lambda$ will be small; and the crowding and spreading of the bands should occur approximately at a maximum or a minimum point, just as in a case when air is used as a means of comparison. For a point of inflection, the test might indicate a close maximum-minimum region. Thus, water (as a solvent), glass, mica, etc., could be used in detecting anomalous dispersion, or a definite number of air-bands may be added.

If, instead of there being a single medium in the path of each of the systems of interfering rays, there are several, in addition to the anomalous medium to be examined, similar considerations will show that the criteria for the anomalies are the same.

Let the resultant orders for the two sets of bands which are to be compared be represented by N' and N'' where $N' = N + N_1' + N_2' + \cdots N_m'$ and $N'' = N_0 + N_1'' + N_2'' + \cdots N_m''$.

Then if

$$\frac{\partial N'}{\partial \lambda} = \frac{\partial N''}{\partial \lambda},$$

we have

$$\frac{\partial N}{\partial \lambda} - \frac{\partial N_0}{\partial \lambda} + \frac{\partial N_1'}{\partial \lambda} - \frac{\partial N_1''}{\partial \lambda} + \cdots \frac{\partial N_m'}{\partial \lambda} - \frac{\partial N_m''}{\partial \lambda} = 0.$$

When the orders of the two sets of interfering bands are made the same before inserting the media to be compared, as can readily be realized in various interfering systems by suitable experimental conditions, we shall have

$$N_1' = N_1'', N_2' = N_2'' \cdots N_m' = N_m''$$

and hence

$$\frac{\partial N}{\partial \lambda} = \frac{\partial N_0}{\partial \lambda} \qquad (9)$$

Thus the crowding and spreading of the resultant bands will indicate the presence of anomalies as before. The additional bands then serve as lines of reference for making measurements, when the orders of the media to be compared are too low to give sufficient bands to make consecutive comparisons throughout the spectrum. The equality $N = N_0$, however, cannot be determined from (3) so readily as before by the process of counting. Either the additional bands must be excluded in order to use (3) as usual, or the bands from these media must be counted when they are introduced into the field from zero order up to the orders to be used. For low orders, less than unity, the latter method is usually preferable. With the previous method we may determine whether we have a true zero order by inserting a compensating element of the order m, say for the D-line. The dispersion of this compensating element must be the same as that of the medium producing the bands. If then we do not find the final band on the D-line, the interfering system may be adjusted until we get coincidence of the order with the D-line. We may then proceed in the usual way, by counting, to obtain the

order of the unknown element, noting, however, any change from the zero order of the interfering system and adjusting, accordingly, if necessary. Should there be any uncertainty as to the proper compensation for other parts of the spectrum, we should note whether the other bands are in their proper places in the rest of the spectrum also. The various characteristics discussed above may be readily represented graphically. From equations (4) we have

$$N\lambda = d\mu, \tag{4a}$$

and

$$N_0\lambda = d_0\mu_0 h, \tag{4b}$$

where μ is a constant. (4*b*) represents a series of equilateral hyperbolæ, with different parameters depending on $d_0\mu_0$ being taken as unity. In Fig. 3 the curve *A B C D E* is plotted from Cartmel's

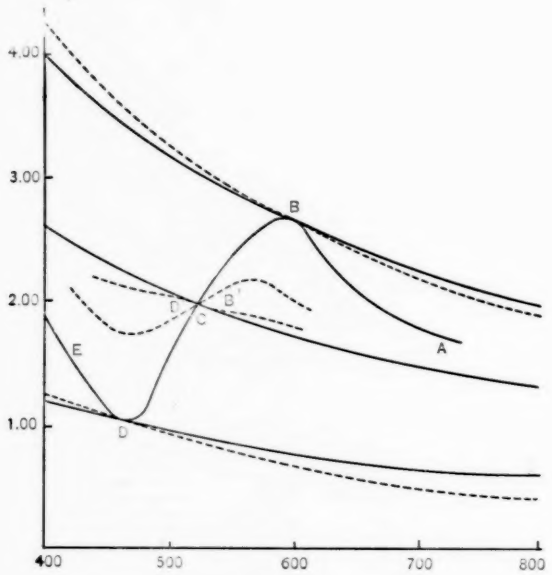

Fig. 3.

data. The full lines represent the rectangular hyperbolæ, while dotted lines represent the deviations from the corresponding hyperbolæ, when μ_0 varies but slowly, as it does considerably beyond the anomalous region. The curve *A B C D E* represents the variations in N due to the fluctuations of μ in its anomalous region. Thus from

A to B, N increases more rapidly than N_0, from B to D more slowly, then more rapidly from D to E. Beyond A, N increases less rapidly than N_0, becoming finally asymptotic to the axis of λ at $\lambda = \lambda_\infty$, while beyond E, N increases less rapidly, becoming asymptotic to the axis of N at $\lambda = \lambda_0$ where μ becomes unity. In the case of an inflexion, the curve $B'CD'$ becomes tangent on crossing the hyperbola C. Thus the order N increases more rapidly before and after crossing the hyperbola, but diminishes more rapidly beyond B' and D'.

Fig. 2 is a reproduction of Cartmel's fuchsin-bands, showing the fluctuations in width referred to. Toward the violet end, we see the fuchsin-bands are relatively narrower, while they are wider than the air-bands in the central portion, indicating a minimum in this region. At the red end the fuchsin-bands are narrower than the air-bands, thus indicating a maximum region.

The method of interferences in the study of dispersion has much to recommend it over the usual prismatic methods. The strong absorption in the anomalous region prevents the use of any but the smallest angled prism, which involves serious instrumental errors. The spectral interference methods referred to eliminate these errors, since the factors sought can be determined by a process of counting and are hence independent of the angle of the film, which may vary in any arbitrary way so long as the change in thickness is not so abrupt as to prevent the "count" of the bands. Media through which it would be impossible to obtain sufficient light for measurements, if in prismatic form, can readily be examined in thin films in this way.

The use of spectral bands presents us with a method which unfolds before the eye, in the position of the bands in the spectrum, an exact record of whatever the medium has impressed upon the various trains of waves which have passed through it. Using the criteria mentioned above, we may at once locate the presence of any dispersive characteristic in the spectrum which the medium may possess. Thus, if in addition a quantitative determination is to be made, we may localize our measurements to this particular region.

The special method of interference to be adopted will depend partly on the nature of the media studied. When the films are thick enough and also sufficiently transparent to give the charac-

teristic interference bands by reflection, when examined directly by the spectroscope, observations are simplified. Comparison can be conveniently obtained by means of a double rectangular collimator one half the field of which is illuminated with light from one of its slits directly, and the other half with light from the other slit after total reflection from a rectangular prism placed before the objective similar to the arrangement used by the writer[1] and later by Rendtorff,[2] in the study of differential double refraction. In this case, total reflecting prisms may be placed at the slits to throw the light upon the films and thence into the collimator, as used by Cartmel[3] in finding the thickness of his films. Instead of the above arrangement, two collimators may be arranged with the compound prism as used in the spectrophotometer of the writer.[4] These arrangements allow of independent movements of each of the two media under examination, and facilitate the adjustments considerably over those made when the two films are placed in juxtaposition before the same slit. The former arrangement also brings the two sets of bands closer together. If the absorption is great, the film to be examined may be placed between semi-transparent mirror-surfaces to equalize the intensities of the two interfering beams. If the order is too low to give sufficient bands for comparison, an auxiliary film may be introduced in series with the film to be examined and the method of comparison of mixed bands described above be used. Under the latter conditions, we may also use an interferometer, preferably of such a type that the light may pass through the film once only. The method adopted by the writer, of converging the light within an interferometer similar in type to that of Jamin and of using a mixed system of bands, should be chosen. Cartmel[5] has used this arrangement most effectively in his studies on the dispersion of fuchsin. Here again we have the spectral bands after passing the interferometer. This spectral analysis of interferential phenomena has much to recommend it over the customary mode of analyzing the light first and examining afterward, with homogeneous rays, these interference effects.

[1] L. c.
[2] L. c.
[3] L. c.
[4] Phil. Mag., Nov., 1899, p. 420; Astrophys. Journ., Jan., 1900.
[5] L. c.

When very slight anomalies are present and the absorption is small, we should resort to high orders and the high resolving power of the grating. Thus, in the examination of weak solutions of varying concentrations of anomalous substances, the growth of the anomaly is at once made evident. Here, several thousand bands may be introduced into the spectrum, and, as the effect is cumulative, any slight relative shift in the bands becomes at once evident. Comparison bands, from the solvent say, may be introduced by means of a second slit arranged in duplication of the system used by Williams[1] in determining refractive indices, the Frauenhofer lines being used for reference along with the bands themselves. In fact, for the study of anomalies in gases, this arrangement should be used on account of the high resolving power required.

Rotary Polarization.

As is well known, if a rotary medium be placed between two nicol prisms and the light transmitted through this system be examined spectrally, dark bands will appear for those wave-lengths the principal planes of whose vibrations are at right angles to that of the analyzing nicol. This extinction will recur for every increment or decrement in the rotation of 180°. Thus the order of the bands will be proportional to the rotation, or

$$\frac{\partial N}{-\partial \lambda} \propto \frac{\partial \rho}{-\partial \lambda}.$$

Hence if

$$\frac{\partial N}{-\partial \lambda} = \frac{\partial N_0}{-\partial \lambda},$$

$$\frac{\partial \rho}{-\partial \lambda} = \frac{\partial \rho_0}{-\partial \lambda}$$

and

$$\rho = \rho_0 + \text{const.} \qquad (10)$$

is the condition for achromatism of such a couple. The total rotation may be obtained by the process of counting, as described above, to obtain the order, the thickness of the rotary medium being varied. Thus, taking black bands as our basis of calculation, the

[1] L. c.

nicols must be crossed. Since here our order corresponds to a rotation equal to 180°, we have from (3)

$$N_D 180° = \rho_D = \frac{m}{n-1} 180°. \qquad (11)$$

The application of this spectral method to achromatic comparisons is practically limited to a very few substances on account of the great rotations necessary to produce sufficient bands.

Thus, quartz would give only about 9 or 10 bands in the visible spectrum for a thickness of 50 mm. As quartz has a much higher rotary power than most active substances, very considerable lengths of the media would be necessary in any comparisons. The same is true of magnetically active media. For this same reason the above tests for anomalous rotary dispersion are impracticable except with mixed systems, using perhaps quartz to give sufficient bands for comparison. Exact quantitative comparison should of course be made by means of half-shade plane polarizing systems. The importance of achromatism to saccharimetry warrants, perhaps, such spectral comparisons, to determine the region in the spectrum giving the best rotary achromatism with quartz; or, in fact to determine a more perfectly achromatizing comparator than quartz itself.

DOUBLE DIFFERENTIAL REFRACTION.

The relative retardation produced by a double-refracting plate placed between two nicols, may also be examined by means of a spectroscope. Thus, if the nicols are crossed and the principal axis of the plate makes an angle of 45° with that of the polarizing system, the light transmitted by such a plate will be cut out when the difference in phase between the two components, expressed in wavelengths, is a whole number. As this difference is a function of the frequency, we should obtain the well known bands in the spectrum corresponding to those frequencies. I have already used the above method[1] of determining the orders of individual plates and have also determined the particular orders to be used in crossed achromatic couples. Rendtorff[2] has also made similar determinations

[1] L. c.
[2] L. c.

using a grating. This method has revealed persistent errors in old determinations of the orders of plates arising from the assumption that, if

$$\frac{\partial N'}{-\partial \lambda} = \frac{\partial N''}{-\partial \lambda},$$

we shall have $N' = N''$, which is seriously in error in the comparison of one crystal with another, as is customary with crystallographers. Thus, if

$$\frac{\partial N'}{-\partial \lambda} = \frac{\partial N''}{-\partial \lambda},$$

we should have

$$N' = N'' + \text{const.}$$

or

$$d'(\mu_e' - \mu_0') = d''(\mu_e'' - \mu_0'') + \text{const.} \qquad (12)$$

where

$$N' = \frac{d'(\mu_e' - \mu_0')}{\lambda} \text{ and } N'' = d'' \frac{(\mu_e'' - \mu_0'')}{\lambda}$$

and μ_e and μ_0 are the two principal indices.

Mouton[1] claimed to have obtained anomalous differential dispersion in selenite. An examination of Rendtorff's bands by means of the maximum-minimum criteria, shows no such fluctuations. That such anomalies are present in other regions may be expected, since the natural frequencies of the two rectangular components are usually different, as exists in the case of tourmaline, Iceland Spar, etc. That such bands may be tested by our criteria is evident from the factors constituting N, which is evidently a mixed system and, hence, conformable to our rules, being a difference instead of a summation system, the one having a slow variation in the dispersion since it is not near its natural frequency. Hence our comparison bands may be bands from thin films, or they may be differential bands from another crystal having no anomaly in this region. The applications of this method in the case of crystals, as well as the experimental arrangements for making the comparisons, may be found in the papers already referred to.

[1] Wien, Ak. Ber., XXXVI. (2), p. 793, 1877.

ACROMATIC COMPENSATION AND INTERFERENCE.

In many of the determinations of the velocities of systems of waves in different media, white light is used in an interferometer and the measurements are made to depend upon the position of the so-called "central" or white band. When a zero method is adopted, an optical compensation is used to bring back this "central" band to the fiducial line. If this is not done the apparent displacement of this band is usually measured directly. Either of these methods may introduce serious errors in the results. Considerations, similar to those mentioned above as to the achromatism of the two systems of interfering rays, will enable us to determine when such errors do enter, and their proper corrections. In the "zero" method, the compensation is produced either by an adjustment of one of the surface elements of the interferometer or by the introduction of an additional amount of a retarding medium. An example of the former practice is shown in the Michelson type and of the latter in the Jamin type of inteferometer. It is evident that both the zero and the displacement methods involve essentially the same chromatic differentiation, since in both we have the introduction of like compensating components. We may therefore discuss them under the same head. Such systems are mixed systems the consideration of which may be reduced to that of two elements, namely, the one to be examined and the other the compensating factor. The compensating medium is thus either a vacuum (or air) or a solid (like glass, mica, etc.) or a liquid (such as the solvent in a solution, etc.). Referring to equations (9), we have now, as the condition of a white band, our equation of reference

$$\frac{\partial N}{-\partial \lambda} = \frac{\partial N_0}{-\partial \lambda}$$

or

$$N = N_0 + k.$$

Thus our so-called white band will be achromatic to a greater or less degree, depending on the extent of the region over which the first equality holds. If then sN_D is the order of the compensating elements $s(N_0+k)_D$ must be the order of effect introduced by the unknown element. The constant k should be determined beforehand

under the same conditions of luminosity, absorption, etc., as those of the compensation. Thus, if the silicate crown glass strips, usually mounted in connection with Jamin's interferometer, were to be used to compensate displacements due to borate flint specimens, errors would enter in, since, in this case, $k \neq 0$. Similar errors would occur in comparing the solution of certain substances with the solvent itself. When the compensation consists of a displacement of one of the surface elements, the compensating medium is usually air (where of course the paths are so small that dispersion does not enter), and we should not have an achromatic band, unless a region of inflection of the refractive index of the medium existed over the most luminous part of the spectrum used, so as to give a nearly constant index. If this condition did not obtain, the so-called central band would be imperfectly achromatized. Stokes[1] has already called attention to the error in the latter instance in determining refractive indices by the displacement of the bands. This phenomena, referred to by Stokes, is, however, accompanied by a certain amount of chromatism of the band, and for this reason the error is readily detected. In the examples mentioned above this error cannot be thus detected. Our only resource is the use of spectral bands, if a preliminary examination of the two elements is not made beforehand. Mention may again be made of the convenience of this system on account of our ability to determine the order by the method of counting. Thus, if we are determining the index (and dispersion) of a gas, it is quite unnecessary to count the total number of bands, on the introduction of the gas into the path of one of the interfering rays. We only need to vary the quantity by a slight amount, in order to get the number of passages m of the bands at any point in the spectrum and note the ratio of n. The same would hold in the case of solids, solutions, etc. The tediousness of many of the old measurements could thus be greatly reduced.

Attention might also be called to the test for anomalies, in such measurements on gases, which would occur near the region of absorption.

The fact that we may have achromatism without the orders being the same, *i. e.*, $k \neq 0$, if

[1] Math. and Phys. Papers, Vol. II., p. 361.

$$\frac{\partial N}{-\partial \lambda} = \frac{\partial N_0}{-\partial \lambda},$$

shows that in the experiments on so-called Talbot's bands we should not obtain them, even when the retardation is introduced in the appropriate way.

ACHROMATIC REFRACTING SYSTEMS.

We may extend to achromatic systems a treatment[1] similar to that which I have already given, in discussing the variation in direction which is produced by any arbitrary system of reflecting and refracting surfaces upon a ray which has been subjected to a change in its velocity or in the phase of its vibration. Consider the refraction of paraxial rays, N' and N'', at a series of surfaces, s_1, $s_2 \cdots s_m$, very near together. The order of the undeviated ray is $N' = N_0' + N_1' + N_2' + \cdots + N_m'$, and that of a ray at a distance δs from the axis is

$$\begin{aligned} N'' = N_0'' + N_1'' + N_2'' + \cdots + N_m'' = N_0' + N_1' + N_2' \\ + \cdots N_m' + \frac{\partial N_0'}{\partial s}\delta s + \frac{\partial N_1'}{\partial s}\delta s + \frac{\partial N_2'}{\partial s}\delta s + \cdots + \frac{\partial N_m'}{\partial s}\delta s. \end{aligned} \tag{13}$$

In all cases of refraction at a surface, destructive interference must occur at all points in space, corresponding to this interval, beyond the surfaces, except at a point on the undeviated ray which is designated as the focus of the system of rays. All disturbances must therefore arrive in the same phase at this point. Hence

$$N' = N''$$

or

$$\frac{\partial N}{\partial s} = \frac{\partial}{\partial s}(N_0 + N_1 + N_2 + \cdots N_m) = 0. \tag{14}$$

This corresponds to the condition of zero spherical aberration.

The order of the axial ray for some other wave-length, $\lambda + \delta\lambda$, is

$$\begin{aligned} N' = N_0' + \frac{\partial N_0'}{\partial \lambda}\delta\lambda + N_1' + \frac{\partial N_1'}{\partial \lambda}\delta\lambda \\ + N_2' + \frac{\partial N_2'}{\partial \lambda}\delta\lambda + \cdots N_m' + \frac{\partial N_m'}{\partial \lambda}\delta\lambda \end{aligned}$$

[1] Phil. Mag., Apr., 1901, p. 464.

and for a ray of the same wave-length at distance δs is

$$N'' = N_0'' + \frac{\partial N_0''}{\partial \lambda}\delta\lambda + N_1'' + \frac{\partial N_1''}{\partial \lambda}\delta\lambda + N_2'' + \frac{\partial N_2''}{\partial \lambda}\delta\lambda + \cdots N_m''$$

$$+ \frac{\partial N_m''}{\partial \lambda}\delta\lambda = N_0' + \frac{\partial N_0'}{\partial s}\delta s + \frac{\partial N_0'}{\partial \lambda}\delta\lambda + \frac{\partial^2 N_0'}{\partial\lambda\partial s}\delta\lambda\delta s$$

$$+ N_1' + \frac{\partial N_1'}{\partial s}\delta s + \frac{\partial N_1'}{\partial \lambda}\delta\lambda + \frac{\partial^2 N_1'}{\partial\lambda\partial s}\delta\lambda\delta s +$$

. . . . . .

$$+ N_m' + \frac{\partial N_m'}{\partial s}\delta s + \frac{\partial N_m'}{\partial \lambda}\delta\lambda + \frac{\partial^2 N_m}{\partial\lambda\partial s}\delta\lambda\delta s.$$

As $N' = N''$ for every wave-length, we must have as before

$$\frac{\partial N}{\partial s} = 0$$

and also

$$\frac{\partial^2 N}{\partial\lambda\partial s} = 0 \quad \text{or} \quad \frac{\partial N}{\partial \lambda} = \text{const.} \tag{15}$$

We shall now transform these equations so as to embody the factors of index, curvature, etc., used in the ordinary treatment of refracting surfaces. Let there be a single refracting surface s_1, Fig. 4, whose radius of curvature is r_1.

Since

$$N = \frac{\mu l}{\lambda},$$

we have

$$\frac{\partial N_0}{\partial s} = \frac{\mu_0}{\lambda}\frac{\partial l_0}{\partial s}. \tag{16}$$

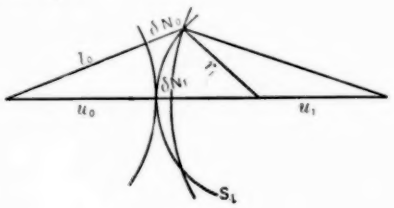

Fig. 4.

By geometry we have also

$$\frac{\partial l}{\partial s} = \frac{\delta s}{2r}. \tag{17}$$

Since we are dealing with paraxial systems in which we neglect second-order small quantities, if we put $\delta s = 2$, we have

$$\frac{\partial l}{\partial s} = \frac{1}{r}. \tag{18}$$

Hence if u_0 is the distance from the radiant to the surface s_1

$$\frac{\partial N_0}{\partial s} = \frac{\mu_0}{\lambda}\left(\frac{1}{u_0} - \frac{1}{r_1}\right). \tag{19}$$

Similarly if u_1 is the distance from s_1 to the image of the radiant,

$$\frac{\partial N_1}{\partial s} = \frac{\mu_1}{\lambda}\left(\frac{1}{r_1} - \frac{1}{u_1}\right). \tag{20}$$

But from (14)

$$\frac{\partial N}{\partial s} = \frac{\partial}{\partial s}(N_0 + N_1) = 0$$

or substituting (19) and (20), we have

$$\frac{\mu_0}{\lambda}\left(\frac{1}{u_0} - \frac{1}{r_1}\right) + \frac{\mu_1}{\lambda}\left(\frac{1}{r_1} - \frac{1}{u_1}\right) = 0 \quad \text{or} \quad \frac{\mu_1}{u_1} - \frac{\mu_0}{u_0} = \frac{\mu_1 - \mu_0}{r_1}. \tag{21}$$

For a vacuum, $\mu_0 = 1$ and we have

$$\frac{\mu_1}{u_1} - \frac{1}{u_0} = \frac{\mu_1 - 1}{r_1}. \tag{22}$$

If there be a single medium, bounded by two surfaces s_1 and s_2, within a second medium, we have similarly

$$\frac{\partial N_0}{\partial s} = \frac{\mu_0}{\lambda}\left(\frac{1}{u_0} - \frac{1}{r_1}\right) \tag{19}$$

$$\frac{\partial N_1}{\partial s} = \frac{\mu_1}{\lambda}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{20}$$

$$\frac{\partial N_2}{\partial s} = \frac{\mu_2}{\lambda}\left(\frac{1}{r_2} - \frac{1}{u_2}\right). \tag{23}$$

From (14),

$$\begin{aligned}\frac{\partial N}{\partial s} = \frac{\partial}{\partial s}(N_0 + N_1 + N_2) = 0 = \frac{\mu_0}{\lambda}\left(\frac{1}{u_0} - \frac{1}{r_1}\right) \\ + \frac{\mu_1}{\lambda}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{\mu_2}{\lambda}\left(\frac{1}{r_2} - \frac{1}{u_2}\right) = 0.\end{aligned} \tag{24}$$

Since $\mu_0 = \mu_2$, we have finally

$$\frac{1}{f_1}=\frac{1}{u_2}-\frac{1}{u_0}=\left(\frac{\mu_1}{\mu_0}-1\right)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

or, if $\mu_0=1$ (25)

$$\frac{1}{f_1}=\frac{1}{u_2}-\frac{1}{u_0}=(\mu_1-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

Consider now two lenses in contact; equation (15) gives us the condition for achromatism for the two wave-lengths λ and $\lambda+\delta\lambda$. We must therefore substitute in (15) the values of $\partial N/\partial s$ given in (19), (20) and (23), and differentiate with respect to λ. Thus for the first lens,

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial N_0}{\partial s}\right)=\frac{1}{\lambda}\left(\frac{1}{u_0}-\frac{1}{r_1}\right)\frac{\partial\mu_0}{\partial\lambda}-\frac{\mu_0}{\lambda^2}\left(\frac{1}{u_0}-\frac{1}{r_1}\right) \tag{26}$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial N_1}{\partial s}\right)=\frac{1}{\lambda}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\frac{\partial\mu_1}{\partial\lambda}-\frac{\mu_1}{\lambda^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \tag{27}$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial N_2}{\partial s}\right)=\frac{1}{\lambda}\left(\frac{1}{r_2}-\frac{1}{u_2}\right)\frac{\partial\mu_2}{\partial\lambda}-\frac{\mu_2}{\lambda^2}\left(\frac{1}{r_2}-\frac{1}{u_2}\right). \tag{28}$$

For the second lens,

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial N_2}{\partial s}\right)=\frac{1}{\lambda}\left(\frac{1}{u_2}-\frac{1}{r_3}\right)\frac{\partial\mu_2}{\partial\lambda}-\frac{\mu_2}{\lambda^2}\left(\frac{1}{u_2}-\frac{1}{r_3}\right) \tag{29}$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial N_3}{\partial s}\right)=\frac{1}{\lambda}\left(\frac{1}{r_3}-\frac{1}{r_4}\right)\frac{\partial\mu_3}{\partial\lambda}-\frac{\mu_3}{\lambda^2}\left(\frac{1}{r_3}-\frac{1}{r_4}\right) \tag{30}$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial N_4}{\partial s}\right)=\frac{1}{\lambda}\left(\frac{1}{r_4}-\frac{1}{u_4}\right)\frac{\partial\mu_4}{\partial\lambda}-\frac{\mu_4}{\lambda^2}\left(\frac{1}{r_4}-\frac{1}{u_4}\right). \tag{31}$$

Making $\mu_0=\mu_2=\mu_4=1$ and substituting the above values in (15), we have

$$\frac{\partial^2 N}{\partial\lambda\partial s}=\frac{1}{\lambda_2}\left\{\lambda\left[\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\frac{\partial\mu_1}{\partial\lambda}+\left(\frac{1}{r_3}-\frac{1}{r_4}\right)\frac{\partial\mu_3}{\partial\lambda}\right]\right.$$
$$\left.+\left[(\mu_1-1)\left(\frac{1}{r_2}-\frac{1}{r_1}\right)+(\mu_3-1)\left(\frac{1}{r_4}-\frac{1}{r_3}\right)+\frac{1}{u_4}-\frac{1}{u_0}\right]\right\}=0. \tag{32}$$

From (24) and (25) we see that we must have

$$\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\frac{\partial\mu_1}{\partial\lambda}+\left(\frac{1}{r_3}-\frac{1}{r_4}\right)\frac{\partial\mu_3}{\partial\lambda}=0 \tag{33}$$

and

$$\frac{1}{u_4}-\frac{1}{u_0}=\frac{1}{f_1}+\frac{1}{f_3}=(\mu_1-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+(\mu_3-1)\left(\frac{1}{r_3}-\frac{1}{r_4}\right). \quad (34)$$

Substituting

$$\left(\frac{1}{r_1}-\frac{1}{r_2}\right)=\frac{1}{f_1(\mu_1-1)}$$

in (33), we have also

$$\frac{1}{f_1(\mu_1-1)}\frac{\partial\mu_1}{\partial\lambda}+\frac{1}{f_3(\mu_3-1)}\frac{\partial\mu_3}{\partial\lambda}=0. \quad (35)$$

From these equations we may calculate the values of $r_1 \cdots r_4$ which will unite the wave-lengths λ and $\lambda+\delta\lambda$ in the conjugate points u_0 and u_4 on the axis.

Substituting in (33) the value of the curvatures from (20), we have

$$\frac{\lambda}{\mu_1}\frac{\partial\mu_1}{\partial\lambda}\frac{\partial N_1}{\partial s}+\frac{\lambda}{3}\frac{\partial\mu_3}{\partial\lambda}\frac{\partial N_3}{\partial s}=0. \quad (36)$$

This shows that $\partial N_1/\partial s$ must be of opposite sign to $\partial N_3/\partial s$ for normal dispersion. Hence the variation in thickness of the one lens must be opposite to that of the other to obtain achromatism.

The value of $(\partial\mu_1/\partial\lambda)/(\partial\mu_3/\partial\lambda)$ from (33) may be readily obtained by means of spectral bands. We have from (5)

$$\begin{aligned}\frac{\partial N_0}{-\partial\lambda}&=\frac{d_0\mu_0}{\lambda^2}\\ \frac{\partial N_1}{-\partial\lambda}&=\frac{d_1\mu_1}{\lambda^2}+\frac{d_1}{\lambda}\frac{\partial\mu_1}{-\partial\lambda}\\ \frac{\partial N_3}{-\partial\lambda}&=\frac{d_3\mu_3}{\lambda^2}+\frac{d_3}{\lambda}\frac{\partial\mu_3}{-\partial\lambda}.\end{aligned} \quad (37)$$

If we make $N_0=N_1=N_3$, we have

$$\frac{d_3}{d_1}\frac{\dfrac{\partial N_1}{-\partial\lambda}-\dfrac{\partial N_0}{-\partial\lambda}}{\dfrac{\partial N_3}{-\partial\lambda}-\dfrac{\partial N_0}{-\partial\lambda}}=\frac{\dfrac{\partial\mu_1}{-\partial\lambda}}{\dfrac{\partial\mu_3}{-\partial\lambda}}. \quad (38)$$

Thus if we have air-bands in juxtaposition with the bands of the first medium, and then with those of the second medium, and adjust until they have the same order for one wave-length, λ_c, say, the ratio of the respective increments in order, over the air-bands for another wave-length, will be the ratio of the increments in indices sought.

In order that m wave-lengths may have a common focus, m lenses would be required, and we should find in a similar way the following $m - 1$ equations,

$$\begin{aligned}
&\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\frac{\partial\mu_1}{\partial\lambda_1}+\left(\frac{1}{r_3}-\frac{1}{r_4}\right)\frac{\partial\mu_3}{\partial\lambda_1}+\cdots+\left(\frac{1}{r_{2m-1}}-\frac{1}{r_{2m}}\right)\frac{\partial\mu_{2m-1}}{\partial\lambda_1}=0\\
&\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\\
&\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\frac{\partial\mu_1}{\partial\lambda_{m-1}}+\left(\frac{1}{r_3}-\frac{1}{r_4}\right)\frac{\partial\mu_3}{\partial\lambda_{m-1}}+\cdots+\left(\frac{1}{r_{2m-1}}-\frac{1}{r_{2m}}\right)\frac{\partial\mu_{2m-1}}{\partial\lambda_{m-1}}=0
\end{aligned}\qquad(39)$$

or

$$\begin{aligned}
&\frac{1}{f_1(\mu_1-1)}\frac{\partial\mu_1}{\partial\lambda_1}+\frac{1}{f_3(\mu_3-1)}\frac{\partial\mu_3}{\partial\lambda_1}+\cdots+\frac{1}{f_{2m-1}(\mu_{2m-1}-1)}\frac{\partial\mu_{2m-1}}{\partial\lambda_1}=0\\
&\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\\
&\frac{1}{f_1(\mu_1-1)}\frac{\partial\mu_1}{\partial\lambda_{m-1}}+\frac{1}{f_3(\mu_3-1)}\frac{\partial\mu_3}{\partial\lambda_{m-1}}+\cdots+\frac{1}{f_{2m-1}(\mu_{2m-1}-1)}\frac{\partial\mu_{2m-1}}{\partial\lambda_{m-1}}=0
\end{aligned}\qquad(40)$$

and the focal equation

$$\begin{aligned}
\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_3}+\cdots+\frac{1}{f_{2m-1}}&=(\mu_1-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\\
&+(\mu_3-1)\left(\frac{1}{r_3}-\frac{1}{r_4}\right)+\cdots+(\mu_{2m-1}-1)\left(\frac{1}{r_{2m-1}}-\frac{1}{r_{2m}}\right).
\end{aligned}\qquad(41)$$

Here $\partial\lambda_1$, $\partial\lambda_2$, $\cdots\partial\lambda_{m-1}$ are the increments in wave-lengths referred to some specified wave-length, λ_D, say; and $\partial\mu_1$, $\partial\mu_3\cdots\partial\mu_{2m-1}$, the corresponding increments in index in each lens.

The above equations show that the order of sequence of the lenses is arbitrary. This method of satisfying the equation of achromatism (15), by achromatizing for different wave-lengths, gives rise to so-called secondary spectra.

We may, however, satisfy this equation of condition in a different way, which can now be realized, in part at least, experimentally,

with less residual chromatism than the old method of using a greater number of lenses.

Instead of using m lenses of different indices, we may combine two having $m+1$ color coincidences. We shall then have the m equations,

$$\begin{aligned}
\frac{\partial^2 N}{\partial\lambda_1\partial s} &= \frac{\partial}{\partial\lambda_1}\left(\frac{\partial N_0}{\partial s}+\frac{\partial N_1}{\partial s}+\frac{\partial N_2}{\partial s}+\frac{\partial N_3}{\partial s}+\frac{\partial N_4}{\partial s}\right)=0\\
\frac{\partial^2 N}{\partial\lambda_2\partial s} &= \frac{\partial}{\partial\lambda_2}\left(\frac{\partial N_0}{\partial s}+\frac{\partial N_1}{\partial s}+\frac{\partial N_2}{\partial s}+\frac{\partial N_3}{\partial s}+\frac{\partial N_4}{\partial s}\right)=0\\
&\vdots\\
\frac{\partial^2 N}{\partial\lambda_m\partial s} &= \frac{\partial}{\partial\lambda_m}\left(\frac{\partial N_0}{\partial s}+\frac{\partial N_1}{\partial s}+\frac{\partial N_2}{\partial s}+\frac{\partial N_3}{\partial s}+\frac{\partial N_4}{\partial s}\right)=0
\end{aligned} \qquad (42)$$

Proceeding in the same way as with (32), we should thus have the m equations

$$\begin{aligned}
\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\frac{\partial\mu_1}{\partial\lambda_1}+\left(\frac{1}{r_3}-\frac{1}{r_4}\right)\frac{\partial\mu_3}{\partial\lambda_1}&=0\\
\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\frac{\partial\mu_1}{\partial\lambda_2}+\left(\frac{1}{r_3}-\frac{1}{r_4}\right)\frac{\partial\mu_3}{\partial\lambda_1}&=0\\
&\vdots\\
\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\frac{\partial\mu_1}{\partial\lambda_m}+\left(\frac{1}{r_3}-\frac{1}{r_4}\right)\frac{\partial\mu_3}{\partial\lambda_m}&=0
\end{aligned} \qquad (43)$$

and the focal equation

$$\frac{1}{f_1}+\frac{1}{f_3}=(\mu_1-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+\mu_3-1\left(\frac{1}{r_3}-\frac{1}{r_4}\right). \qquad (34)$$

We have seen how, by means of (37) and (38), to achromatize for two wave-lengths by means of two lenses. We may by means of (43) achromatize for $m+1$ wave-lengths by means of *two* lenses. Suppose $m=2$, we have then to find the corresponding equations similar to (38) in terms of their orders. We have from (5)

$$\begin{aligned}
\frac{\partial N_0}{-\partial\lambda_1} &= \frac{d_0\mu_0}{\lambda^2}\\
\frac{\partial N_1}{-\partial\lambda_1} &= \frac{d_1\mu_1}{\lambda^2}+\frac{d_1}{\lambda}\frac{\partial\mu_1}{-\partial\lambda_1}\\
\frac{\partial N_1}{-\partial\lambda_2} &= \frac{d_1\mu_1}{\lambda^2}+\frac{d_1}{\lambda}\frac{\partial\mu_1}{-\partial\lambda_2}
\end{aligned} \qquad (44)$$

$$\frac{\partial N_3}{-\partial \lambda_1} = \frac{d_3 \mu_3}{\lambda^2} + \frac{d_3}{\lambda} \frac{\partial \mu_3}{-\partial \lambda_1}$$

$$\frac{\partial N_3}{-\partial \lambda_2} = \frac{d_3 \mu_3}{\lambda^2} + \frac{d_3}{\lambda} \frac{\partial \mu_3}{-\partial \lambda_2}$$

where $\partial \lambda_1$ and $\partial \lambda_2$ are the increments in wave-length, referred to some wave-length of reference λ_D, of any two other wave-lengths, λ_C and λ_F, say.

If we make $N_0 = N_1 = N_3$, we shall have

$$\frac{\dfrac{\partial N_1}{-\partial \lambda_1} - \dfrac{\partial N_0}{-\partial \lambda_1}}{\dfrac{\partial N_1}{-\partial \lambda_2} - \dfrac{\partial N_0}{-\partial \lambda_2}} = \frac{\dfrac{\partial \mu_1}{-\partial \lambda_1}}{\dfrac{\partial \mu_1}{-\partial \lambda_2}} \tag{45}$$

$$\frac{\dfrac{\partial N_3}{-\partial \lambda_1} - \dfrac{\partial N_0}{-\partial \lambda_1}}{\dfrac{\partial N_3}{-\partial \lambda_2} - \dfrac{\partial N_0}{-\partial \lambda_2}} = \frac{\dfrac{\partial \mu_3}{-\partial \lambda_1}}{\dfrac{\partial \mu_3}{-\partial \lambda_2}}. \tag{46}$$

But (43) requires that

$$\frac{\dfrac{\partial \mu_1}{-\partial \lambda_1}}{\dfrac{\partial \mu_1}{-\partial \lambda_2}} = \frac{\dfrac{\partial \mu_3}{-\partial \lambda_1}}{\dfrac{\partial \mu_3}{-\partial \lambda_2}} \tag{47}$$

and hence we must have

$$\frac{\dfrac{\partial N_1}{-\partial \lambda_1} - \dfrac{\partial N_0}{-\partial \lambda_1}}{\dfrac{\partial N_3}{-\partial \lambda_1} - \dfrac{\partial N_0}{-\partial \lambda_1}} = \frac{\dfrac{\partial N_1}{-\partial \lambda_2} - \dfrac{\partial N_0}{-\partial \lambda_2}}{\dfrac{\partial N_3}{-\partial \lambda_2} - \dfrac{\partial N_0}{-\partial \lambda_2}}. \tag{48}$$

Equations (34) and (38) are the conditions for achromatism of two colors, while (48) is the further condition for achromatism of three colors by means of two lenses. Thus, suppose we have our three sets of spectral bands in the field of view, the air-bands being in the middle, we set for coincidence at the D-line say, making the sets of the same order, which can readily be accomplished by varying the thicknesses successively and counting the bands, as described above. The eye may then note the gain relatively to the air-bands of each

of the other sets of bands on each side of the D-line. If these two ratios should become equal at two points on each side of the D-line, we know that we shall have chromatic compensation for all three colors. Should we fail to find this condition satisfied, we may shift the region of coincidence with the air-bands to some other point and examine for a closer approximation of the ratios, and so on. Very close approximations to such ratios are now possible with certain combinations of the silicate and borate glasses, such as those obtainable from the Jena factory. Compare, for example, factory Nos. 0.152 and S.7, whose partial dispersions for $A' - D$, $D - F$, $F - G'$ are .628, .708, .582 and .628, .708 and .583 respectively. The facility with which comparisons may be made with different experimental castings in obtaining new series and pairs of glasses is evident from the above discussion.

The treatment for media with plane surfaces, *i. e.*, achromatic prisms, is similar to that for curved surfaces or lenses. Thus in the case of two thin prisms, we have to satisfy equations (14) and (15). We obtain, as before,

$$\begin{aligned} \frac{\partial N_0}{\partial s} &= \frac{\mu_0}{\lambda}\frac{\partial l_0}{\partial s} = \frac{\mu_0}{\lambda}\tan i = \frac{\mu_0}{\lambda} i \\ \frac{\partial N_1}{\partial s} &= \frac{\mu_1}{\lambda}\frac{\partial l}{\partial s} = -\frac{\mu_1}{\lambda}\tan a = \frac{\mu_1}{\lambda} a_1 \\ \frac{\partial N}{\partial s} &= \frac{\mu}{\lambda}\frac{\partial l}{\partial s} = \frac{\mu}{\lambda}\tan i' = \frac{\mu}{\lambda} i'. \end{aligned} \tag{49}$$

where i is the angle of incidence and a that of the prism. From (14) we have

$$\frac{\partial N}{\partial s} = \frac{\partial N_0}{\partial s} + \frac{\partial N_1}{\partial s} + \frac{\partial N}{\partial s} = \frac{\mu_0}{\lambda} i - \frac{\mu_1}{\lambda} a_1 + \frac{\mu}{\lambda} i' = 0.$$

Making $\mu_0 = \mu = 1$, we have, calling D the deviation,

$$i + i' - \mu_1 a_1 = 0 \tag{50}$$

or, since $D = i + i' - a$, we must have

$$D = (\mu_1 - 1)\, a_1. \tag{51}$$

Similarly for two prisms we should find

$$i_1 + i_1' - \mu_1 a_1 + i_3 + i_3' - \mu_3 a_3 = 0, \tag{53}$$

or

$$D = (\mu_1 - 1)\, a_1 + (\mu_3 - 1)\, a_3. \tag{54}$$

Differentiating (49), we have

$$\begin{aligned}
\frac{\partial^2 N_0}{\partial\lambda\partial s} &= \frac{i_1}{\lambda}\frac{\partial\mu_0}{\partial\lambda} - \frac{\mu_0 i_1}{\lambda^2} \\
\frac{\partial^2 N_1}{\partial\lambda\partial s} &= -\frac{a_1}{\lambda}\frac{\partial\mu_1}{\partial\lambda} + \frac{\mu_1 a_1}{\lambda^2} \\
\frac{\partial^2 N_2}{\partial\lambda\partial s} &= \frac{i'}{\lambda}\frac{\partial\mu_2}{\partial\lambda} - \frac{\mu_2 i_1'}{\lambda^2} \\
\frac{\partial^2 N_2}{\partial\lambda\partial s} &= \frac{i_3}{\lambda}\frac{\partial\mu_2}{\partial\lambda} - \frac{\mu_2 i_3}{\lambda^2} \\
\frac{\partial^2 N}{\partial\lambda\partial s} &= -\frac{a_3}{\lambda}\frac{\partial\mu_3}{\partial\lambda} + \frac{\mu_3 a_3}{\lambda^2} \\
\frac{\partial^2 N}{\partial\lambda\partial s} &= \frac{i_3'}{\lambda}\frac{\partial\mu_4}{\partial\lambda} - \frac{\mu_4 i_3'}{\lambda^2}.
\end{aligned} \tag{55}$$

If we make $\mu_0 = \mu_2 = \mu_4 = 1$, we still have

$$\begin{aligned}
\frac{\partial^2 N}{\partial\lambda\partial s} = -\frac{1}{\lambda^2}\Bigg\{ \lambda\left[a_1\frac{\partial\mu_1}{\partial\lambda} + a_3\frac{\partial\mu_3}{\partial\lambda}\right] + [i_1 + i_1' + i_3 \\
+ i_3' - \mu_1 a_1 - \mu_3 a_3]\Bigg\} = 0
\end{aligned} \tag{56}$$

or from (53)

$$a_1\frac{\partial\mu_1}{\partial\lambda} + a_3\frac{\partial\mu_3}{\partial\lambda} = 0. \tag{57}$$

It is evident from this equation that the angles a_1 and a_3 must have opposite signs. We may proceed to a higher order of achromatism by using a greater number of prisms, just as with lenses, obtaining equations similar to (39) and (40). Or we may reduce the residual chromatism in a pair of prisms by satisfying the equation (48) in addition to (53) and (57).

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THE RATE OF RECOMBINATION OF THE IONS IN AIR.

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IF a gas is ionized by any ionizing agent such as Röntgen or radium rays the positive and negative ions will immediately begin to recombine due to their motion and mutual attractions. A simple law, first suggested by J. J. Thomson and Rutherford[1] and obtained by analogy from the law of mass action in physical chemistry, has been found to cover this phenomenon. It states that the rate of disappearance of either charge is proportional to the positive and negative electrical densities, p and n. Thus

$$\frac{dp}{dt} = \frac{dn}{dt} = -\alpha np \qquad (1)$$

where the proportionality factor α is called the coefficient of recombination.

If p and n represent the number of ions per c.c. instead of the charges on these ions the coefficient will be the α above times e, the charge on one ion.

Rutherford,[2] McClung[3] and Langevin[4] have investigated this law with the conclusion that it is a true one, while Townsend,[5] McClung and Langevin have determined the value of the coefficient α for several gases, agreeing with each other as to its value well within the experimental error at atmospheric pressure. For the variation of α with the pressure however, which has been taken up for air by McClung and Langevin, the agreement has not been so good; McClung coming to the conclusion that the coefficient of recombi-

[1] Phil. Mag., V., 2, p. 92, 896.
[2] Phil. Mag., Nov., 1897, p. 44; Jan., 1899.
[3] Phil. Mag., March, 1902.
[4] Thèse prèsentée à la Facµlté des Sciences.
[5] Phil. Trans. Roy. Soc., p. 157, 1899.

nation is independent of the pressure, while Langevin obtained a rapid decrease.

The purpose of the present investigation was to obtain additional data as to the variation of the coefficient of recombination with the pressure, also to obtain values below the limits of pressure previously used.

The results of the work given in a later paragraph show that α decreases with the pressure, though not so rapidly as Langevin's results would indicate. Determinations were made down to 10 mm. pressure as compared with 100 mm. and 150 mm., the limits reached by McClung and Langerin respectively. Values of the coefficient of diffusion of the ions were also obtained at 20 and 10 mm. pressure.

Method of McClung.

Calling q the total charge on the ions, positive or negative, produced per second by the ionizing agent in each c.c. of the gas, and assuming that there is an equal number of positive and negative ions in each c.c.

$$\frac{dn}{dt} = q - \alpha n^2. \tag{2}$$

Thus after the ionization begins the number of ions in the gas will constantly increase until a state of equilibrium is reached between the rate of recombination αn^2 and the rate of production q, *i. e.*, until $dn/dt = 0$, when we have the relation

$$\alpha = \frac{q}{n^2}, \tag{3}$$

or if Q and N represent the volume integrals of q and n in a certain volume V

$$\alpha = \frac{Q}{N^2} K, \tag{4}$$

where K is a constant depending upon the dimensions of the vessel used as an ionization chamber, and determined by the distribution of the ionization. If the ionization is uniform $K = V$. Using this formula McClung made a direct determination of α. His ionizing agent was the Röntgen rays which ionized the gas contained in a

cylindrical vessel 20 cm. long and separated into ten compartments by thin aluminum partitions. The alternate partitions were connected to each other and acted as electrodes. The Q above, which is the saturation current, was determined in the usual way by noting, by means of the movement of an electrometer needle, the rate of charge of a known capacity connected in parallel with the electrometer and one set of electrodes, against which the ions were being driven by a saturation potential.

The instantaneous value of the free charge N was measured by allowing the Röntgen rays to ionize the gas until a steady state was reached, then stopping the ionization by breaking the primary circuit of the induction coil, and simultaneously with the stopping of the rays, throwing on a saturation potential across the electrodes, thus driving all the ions present in the gas against the electrodes. The charge N on these ions was obtained by noting on an electrometer the potential to which a known capacity was raised.

Method of Langevin.

Langevin's experimental determination of α is based on a very important theory in which he proves that practically all of the recombination of the ions can be accounted for by the consideration of their mutual attractions. On this basis he obtains the formula

$$\alpha = 4\pi(k_1 + k_2)\varepsilon$$

in which $(k_1 + k_2)$ is the sum of the specific velocities of the ions and ε is the fraction of the whole number of collisions which result in recombination. The $k_1 + k_2$ was measured directly while ε was obtained from experiments based on a further consideration of the theory of ionization. The velocities of the ions increase with decreasing pressure and the ε, due to the large kinetic energy of the ions for long free paths, decreases, thus theoretically it cannot be predicted whether α should decrease with decreasing pressure or not. Langevin as stated before found a rapid decrease.

Method and Description of the Apparatus.

The method adopted in the present case is similar to that of McClung's in that it depends upon the formula

$$\alpha = \frac{Q}{N^2 K},$$

but differs from his in that radium was used as the ionizing agent and the N was determined in a new way. For the determination of N, the charge on the ions present in the gas at any time after a steady state has been reached, it was not attempted to stop the ionization, but the electromotive force was allowed to act across the electrodes for varying intervals of time and the charges thus obtained were corrected for the rate of production during that time.

By the use of radium a great ionization compared with that possible from Röntgen rays can be obtained. Also the very serious difficulty of the inconstancy of the radiation, encountered by other observers in the use of Röntgen rays, is avoided, as the intensity of the radium radiations are very constant after the radium has reached a steady state. The chief objection to its use is the error brought in by the uneven distribution of the ionization due to the absorption of the rays. This will be discussed later.

The diagrammatic arrangement of the apparatus is shown in Fig. 1.

A and B are two aluminum plates, respectively 26 and 30 cm. in diameter, supported and insulated by hard rubber and mounted inside a large glass bell-jar. Both A and B were coated as uniformly and equally as possible with a solution of very active radium chloride, about one milligram of the salt being in the solution. In a majority of the experiments A and B were 6 cm. apart. An aluminum ring G, 1.5 cm. wide, was placed around A with .5 cm. space between. This ring was also coated with radium and acted as a guard ring, tending to make the ionization of the volume of gas under the plate A more uniform with respect to the plane of the plates. G was kept grounded so that only the ions directly under A would be driven into A by a potential on B. Those ions contained in the volume extending out half way between A and G were assumed to be caught by A, thus making the effective area 57.2 sq. cm. The upper plate A was connected through the key K_2 to the variable known capacity C and this was connected through the key K_1 in parallel with the electrometer, which was of the

Dolazalek type. For the purposes of the experiment it was necessary to be able to apply an electromotive force to the electrode B for varying intervals of time. To accomplish this the plate B was connected in series with a very high resistance to one terminal of a variable high potential source, the other end of which was grounded. A second circuit was run from B to the earth through the break-circuit key T_1 and the make circuit key T_2. Both T_1 and T_2 were

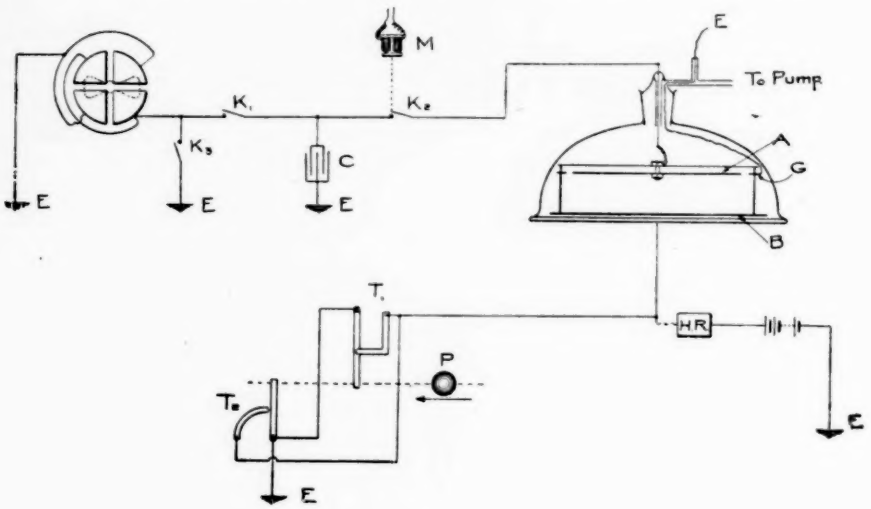


Fig. 1.

placed on adjustable mounts in the path of a pendulum swinging in the path indicated by the arrow. By this arrangement when either T_1 or T_2 was closed B was at zero potential and at the potential of the battery for T_1 and T_2 both open. The pendulum arrangement was calibrated so that the electromotive force could be thrown across B–A for intervals of time varying from .1 to .4 second.

METHOD OF OBSERVATION.

The saturation current Q was obtained in the usual way by noting on the electrometer the rate of increase of the potential of the known capacity C with the time. A correction was applied to allow for the leak from the plate A to the guard ring. This correction was not large as A was never charged to more than one half a volt.

Care was always taken to have a sufficient potential across B–A to drive out the ions as fast as they were produced, thus getting a true saturation current. Due to the large ionization at high pressures, the saturation potential was very high, 6,000 volts being necessary at atmospheric pressure for the plates 6 cm. apart. The method of experimenting for the determination of N, the amount of free electricity on the ions after the gas has reached a steady state was as follows: First, the capacity C was placed at a suitable value, the three keys, K_3, K_2 and K_1 were closed, as was also the break contact key T_1, thus grounding the system. T_2 was left open and a distance corresponding to .1 sec. left between T_1 and T_2. The pendulum was now swung and at the same time the key T_1 was automatically opened, insulating the capacity and electrode A. When the pendulum opened T_1 the ground was removed and the potential of the battery was across B–A for .1 sec., until T_2 was closed. Thus a certain charge was thrown into the capacity, consisting of the quantity of free charge N_1 on the ions in the gas plus the amount on the ions produced by the radium radiation during .1 sec. Simultaneously with the closing of the key T_2, another circuit, not shown in the figure, was closed, thus actuating the electromagnet M which opened the key K_2 and so insulated the capacity C containing the charge. This was done in order to make the leak back through the ionized gas as small as possible. The time from the closing of T_2 to the opening of K_2 was about .1 sec. The charge on C was now measured by closing the key K_1 and noting the deflection on the electrometer. The operation was now repeated with the potential across B–A for .2 sec., giving a second charge consisting of the N as before plus the charge produced for .2 sec. Thus by taking several successive equal intervals of time, throws were obtained which when carried back to zero time gave the throw corresponding to N. A complete set of such readings for air at 100 mm. pressure is given in Table 1 and the resulting curve by the dotted line in Fig. 2. The throws as indicated on the electrometer need two corrections; one for the loss of charge due to the leak back through the ionized gas during the .1 sec. elapsing between the closing of T_2 and the opening of K_2; the second for the leak from the plate A to the guard ring, which begins to come in

when T_1 is opened and continues with increasing value until K_2 is opened. The amount of these corrections depends upon the potential to which C and A are charged and are therefore larger for the longer intervals. The full line in Fig. 2 shows the curve corrected for these two errors. It is seen there that although the corrections may be quite large they effect the value obtained for N comparatively little, merely increasing the slope of the line. For the given pressure, 100 mm. these errors were nearly at a maximum, being

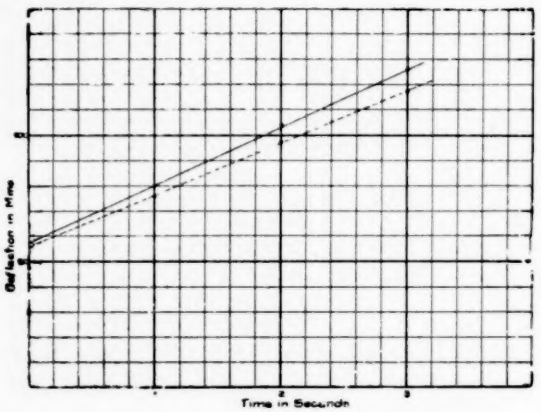


Fig. 2.

much smaller at higher pressures and not so very much larger at the smallest pressures used.

By this method of obtaining N, all disturbances, such as leaks across the electrodes, irregularities in the ionization due to secondary actions of the radiation at the plates or other causes, were eliminated, as they tended merely to effect the slope of the line and not the place where it cuts the axis. The regularity of the readings is shown by the example in Table I. These readings could be repeated at any time fairly closely.

TABLE I.

Illustrative Set of Readings.

Pressure 100 mm. Distance between plates 6 mm. Volume 3435 c.c. Sensitiveness of electrometer 224 divisions for 1 volt on quadrants and 52 volts on needle.

For the saturation current Q: Capacity = .9 microfarads. Volts across plates 440. Time for 30 div., 14 secs. Leak to guard ring 1.6 divs. in 14 secs.

$Q = 31.6$ divs. in 14 secs. $= 27.2$ E. S. units per sec.
For N: Sensitiveness, 224 divs. for 1 volt. Capacity .0102 M.F.

Time E.M.F. Acted Across Plates (Secs.).	.1	.2	.3
Scale divisions.	76	97	117
	77	98	118
	76	96.5	119
	78	99	116.4
	76	97	116
Average.	76.6	97.5	117.3
Leak back through ionized gas.	2.5	3.2	3.9
Leak to guard ring.	.8	2.4	4.4
Corrected average.	79.9	103.1	125.6

Rate of leak back through ionized gas 32 divs./sec. at 100 divs. deflection. Time of leak .1 sec. Rate of leak to guard-ring 12 divs./sec. at 100 divs. deflection.

$\therefore N$ (see Fig. 2) $= 57.2$ divs. $= 7.76$ E. S. units.

$$a = \frac{Q}{N^2} V = 1{,}550.$$

Care was taken to have the air dry and dust-free by passing it through phosphorus pentoxide and cotton wool before letting it into the pump and vessel. Also the question of a saturation potential for N had to be carefully looked into, as it was found that it took a higher potential to insure a maximum throw for N than it did for Q at the same pressure.

Distribution of the Ionization.

It is evident that with a certain rate of production the number of ions in the gas between the plates for a steady state will depend upon the distribution of the ionization. For by equation (1), the number of ions disappearing per sec. in one c.c. varies as the square of the number present, therefore a steady state will be reached with a smaller value of N for large ionization gradients than for small ones. To take this into account it is necessary to know the law of the distribution of the ions.

Rutherford[1] has shown experimentally that the intensity of the ionization due to a large plane surface of active matter, such as radium or uranium, falls off approximately according to an expo-

[1] Phil. Mag., Jan., 1899; "Radioactivity," p. 56.

nential law with the distance from the plates, *i. e.*, calling q the rate of production of the ions per c.c. at a point distant y from the active plate

$$q = q_0 e^{-\lambda y},$$

where q_0 is the rate of production at the surface and λ the absorption constant. This result is deduced theoretically on the assumption that the ionization is proportional to the intensity of the radiation and that the energy of the rays is used up in producing ions. Bragg[1] later on showed that this exponential law is not necessarily a true one, but might hold very closely for the complex radiations from radium spread over a large surface.

In the present case we have rather complex conditions, the ionization being due chiefly to the α rays from the radium on the two plates, but the β rays and the radiation from the emanation throughout the volume of the gas also play some part. To obtain the actual distribution the exponential law was assumed and experiments made to find out how closely it held and for what value of λ.

Taking the origin of coördinates half way between the plates which are $2a$ cm. apart, and the axis x at right angles to the plane of the plates, we have by the exponential law

$$q = q_0 e^{-\lambda(a-x)} + q_0 e^{-\lambda(a+x)} \tag{5}$$

each factor being the contribution of one plate to the ionization q at the point x.

Taking the volume integral of q

$$Q = \int q dv = A \int_{-a}^{+a} q_0 [e^{-\lambda(a-x)} + e^{-\lambda(a+x)}] dx,$$

A being the area of one plate.

Integrating,

$$Q = 2Aq_0 \frac{1 - e^{-2\lambda a}}{\lambda''}. \tag{6}$$

Using this equation and obtaining saturation currents for different distances between the plates it was found that the conditions were closely satisfied for the value $\lambda = .05$, a being expressed in

[1] Phil. Mag., Dec., 1904.

millimeters. This value of λ was obtained by taking any two distances with the corresponding Q's, giving two equations of the form of (6) from which q_0 was eliminated. Knowing λ and starting with the Q for 60 mm. distance the other Q's can be calculated. These values are shown in Table II.

TABLE II.

Pressure 760 *mm.*

Distance Between Plates.	Q Observed.	Q Calculated for $\lambda = .05$.
60 mm.	106	106
30	87	86.6
20	70.8	70.8
10	45.6	44

The constant λ has been found by Rutherford[1] to vary approximately inversely as the density of the absorbing matter. Therefore calling λ_1 the constant at a pressure p_1 and λ its value at 760 mm., we can write eq. (6), since q_0 also varies as the pressure

$$Q_1 = 2Aq_0' (1 - e^{-2a\lambda p_1/760}) \qquad (7)$$

Here Q_1 is the saturation current at the pressure p_1 and q_0' the rate of production per c.c. at the radiating surface at 760 mm. pressure. This equation expresses the variation of the saturation current with the pressure of the gas. Observed values of this variation, also values calculated for $\lambda = .05$, are given in Table III.

TABLE III.

Plates 60 *mm. apart.* $a = 30$ *mm.*

Pressure.	Q Observed.	Q Calculated for $\lambda = .05$.
760	106	106
600	101	101.5
450	94	94
300	77	78
200	55	60.7
100	28.2	36.7
50	14.2	

[1] "Radioactivity," p. 137.

Fig. 3 shows these values plotted. The full line curve shows the observed variation of Q with the pressure, the dotted line the calculated variation. At high pressure the coincidence is very close. Thus we are justified in assuming that at high pressures at least the distribution is given by the exponential law with .05 as the value of the absorption constant. Below 300 mm. pressure the curve is nearly a straight line and the ionization is practically uniform, and so we need no correction.

The value of $\lambda = .05$ obtained here is much smaller than the value

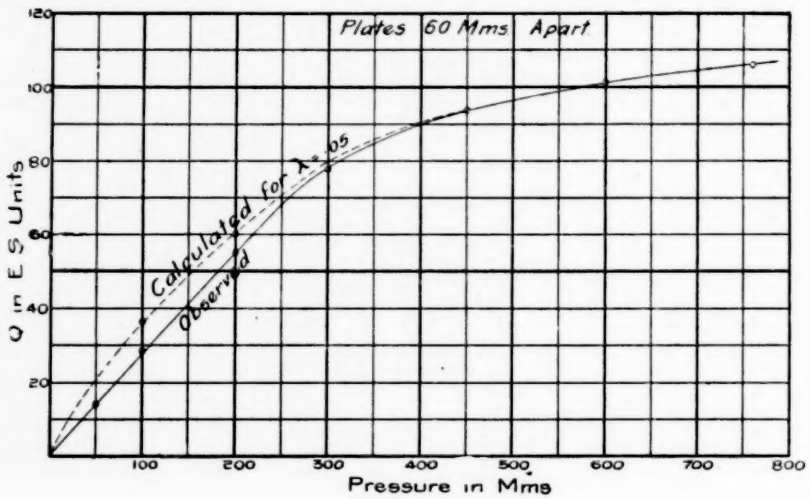


Fig. 3.

.9 given by Rutherford[1] as the absorption constant for the α-rays from radium. His value, however, was obtained by a somewhat different method, under different experimental conditions from the ones used here. In his work radiferous barium was used and the ionization measured after a good part of the radiation had been absorbed by passing through air and aluminium, thus getting a portion of the more easily absorbed α-rays. Also the curves given by Rutherford[2] for variation of the saturation current with the pressure between two large parallel plates, one of which was active, fall

[1] "Radioactivity," p. 137.

[2] Phil. Mag., Jan., 1899, pp. 130, 138.

off much more rapidly than his theory would indicate, using such a large value of λ.

CORRECTION TO a FOR UNEVEN DISTRIBUTION OF THE IONIZATION.[1]

Suppose n as before to be the electric density of the free charge at the point x when a steady state has been reached. Taking the volume integral

$$N = \int n dv,$$

but by equations (3) and (5)

$$n = \sqrt{\frac{q}{\alpha}} = \sqrt{\frac{q_0}{\alpha}(\epsilon^{-\lambda(a-x)} + \epsilon^{-\lambda(a+x)})},$$

therefore

$$N = A\left(\frac{q_0}{\alpha}\right)^{\frac{1}{2}} \int_{-a}^{a} [\epsilon^{-\lambda(a-x)} + \epsilon^{-\lambda(a+x)}]^{\frac{1}{2}} dx.$$

But as the ionization is symmetrical out from the middle point toward either plate the lower limit can be changed from $-a$ to 0 if the integral is multiplied by the factor 2. Doing this, and rewriting

$$N = 2A\left(\frac{q_0}{\alpha}\right)^{\frac{1}{2}} \epsilon^{-\frac{1}{2}\lambda a} \int_{0}^{a} \epsilon^{\frac{1}{2}\lambda x}(1 + \epsilon^{-2\lambda x})^{\frac{1}{2}} dx.$$

Expanding $(1 + \epsilon^{-2\lambda x})^{\frac{1}{2}}$ by the Binomial Theorem, multiplying by $\epsilon^{\frac{1}{2}\lambda x}$, integrating the series term by term and simplifying we have

$$N = 2A\left(\frac{q_0}{\alpha}\right)^{\frac{1}{2}} \frac{1}{\lambda}\left[(4 - \tfrac{2}{3}\epsilon^{-2\lambda a} + \tfrac{1}{14}\epsilon^{-4\lambda a} + +) - \epsilon^{-\frac{1}{2}\lambda a}(4 - \tfrac{2}{3} + \tfrac{1}{14} + +)\right].$$

Since in these experiments λ is .05 and a 30, λa has the value .15, and both of these series are rapidly converging, therefore a close approximation is obtained by neglecting all except the first two terms of the first series and taking 3.38 as the sum of the second. Thus

$$N = 2A\left(\frac{q_0}{\alpha}\right)^{\frac{1}{2}} \frac{4 - \frac{2}{3}\epsilon^{-2\lambda a} - 3.38\epsilon^{-\frac{1}{2}\lambda a}}{\lambda}.$$

[1] See McClung's article, loc. cit.

But from equation (6)

$$q_0 = \frac{\lambda Q}{2A(1 - e^{-2\lambda a})}.$$

Substituting this value and solving for a

$$a = \frac{Q}{N^2} A \frac{(4 - \frac{2}{3}e^{-2\lambda a} - 3.38e^{-\frac{1}{2}\lambda a})^2}{2\lambda(1 - e^{-2\lambda a})}.$$

The exponential term in this equation should be equal to 6, the number of cm. between the plates for uniform ionization. For the value $\lambda a = .15$ this term becomes equal to 5.9. Therefore even at atmospheric pressures the correction for uneven distribution is very small and a can be taken equal to QV/N^2 where V is the volume of gas between the plates.

Effect of Diffusion at Low Pressures.

At above 100 mm. pressure it was found that with the plates 6 cm. apart the loss of ions due to diffusion to the plates and sides of the vessel was negligibly small compared with the loss due to recombination proper. Below this pressure however the observed N began to decrease very rapidly due to diffusion and

$$\frac{Q}{N^2} V$$

to increase correspondingly. It is easily seen how this would come about, since if the loss of ions due to diffusion is appreciable a steady state will be reached for a smaller value of N than it would if there was no such loss.

Due to diffusion there will be an ionization gradient along x from the plane midway between the plates out to their surfaces, where there will be no ionization, the ions there present giving up their charges to the plates. The charge on the ions crossing unit area in unit time due to the gradient dn/dx will be $D\,dn/dx$, where D is the coefficient of diffusion. Therefore the rate of increase of free charge in any unit volume will be $D\,d^2n/dx^2$. Thus for a steady state

$$q = an^2 - D\frac{d^2n}{dx^2}.$$

A satisfactory solution of this equation is not easily found, but for the special case where the factor due to recombination is negligibly small, J. J. Thomson[1] gives the solution

$$n = \tfrac{1}{2}\frac{q}{D}(a^2 - x^2),$$

from which

$$N_0 = \tfrac{1}{3}\frac{qa^2}{D}, \qquad (8)$$

$2a$ being the distance between the plates.

This N_0 represents the charge on the ions which would be in the gas for a steady state if the ions disappeared by diffusion only. The coefficient of diffusion D has been shown by Townsend[2] to vary inversely as the pressure down to 200 mm. so that the loss due to diffusion increases as the pressure decreases. Also it is seen that the diffusion effect varies inversely as the square of the distance between the plates.

This rapid increase of the diffusion effect as the plates get close together was used to get an idea of the error in the observed N brought in by diffusion.

A series of determinations of N and Q were made for different distances between the plates. These results are shown for pressures of 50, 20 and 10 mm. in tables IV.–VI. The fourth column of these tables show the very rapid rise, with the small distances, of the quantity $Q/N^2 V$ which would be a, the coefficient of recom-

TABLE IV.

Area of Plate, 572 *sq. cm. Pressure*, 50 *mm.*

Distance Between Plates.	Q.	N.	$\frac{Q}{N^2} V = a'$.	N_D.
8 cm.	16	7.4	1,340	160
6	14.2	5.9	1,380	80
3	7.64	2.9	1,570	16
2	5.7	1.9	1,830	4

Limiting value of a' is $a = 1{,}310$.

[1] J. J. Thomson, "Conduction of Electricity through Gases," p. 20.

[2] Townsend, Phil. Trans., A, 195, p. 2 9, 1900.

TABLE V.

Pressure, 20 *mm.*

Distance Between Plates.	Q.	N	$\frac{Q}{N^2} V = a'$.	N_D.
8 cm.	6.4	4.75	1,290	25.6
6	4.7	3.14	1,600	10.5
3	3.0	1.14	4,000	1.7
2	2.3	.52	10,000	.58

Limiting value of $a' = a = 1{,}150$.

TABLE VI.

Pressure, 10 *mm.*

Distance Between Plates.	Q.	N.	$\frac{Q}{N^2} V = a'$.	N_D.
8 cm.	3.1	2.7	2,100	6.2
6	2.4	1.4	4,000	2.7
3	1.6	.38	19,000	.45
2	1.21	.143	68,000	.15

Limiting value of $a' = a = 1{,}000$.

bination, if all the loss of ions was considered due to recombination. It is evident that if there was no diffusion, this quantity $Q/N^2 V$, which will be called the apparent a and denoted by a' would be independent of the distance between the plates.

To obtain a at each pressure a curve was plotted between a' as ordinate and the distances between the plates as abscissa, and the value of a' for which this curve became a straight line parallel to the horizontal axis was taken as the true coefficient of recombination. These curves are shown in Fig. 4.

At 50 mm. pressure it is seen that diffusion makes very little error in a even when the plates are 2 cm. distance apart, while at 10 mm. pressure for the small distances between plates it has a very great effect, the loss of ions due to diffusion evidently exceeding that due to recombination. The values of a finally adopted and given in Tables IV.–VI. were taken from these curves plotted on a much larger scale.

A good idea of the way in which diffusion and recombination enter at the different pressures and distances may be obtained by a

consideration of the experimental values of N in the Tables IV.–VI. and the values given in the last column headed N_D. This N_D is the value of N which would have been expected if the loss of the ions balancing the rate of production Q was due to diffusion only. They are calculated from equation (8) taking the coefficient of diffusion D at Townsend's value of .035 for atmospheric pressure and assuming it to vary inversely with the pressure. The larger the

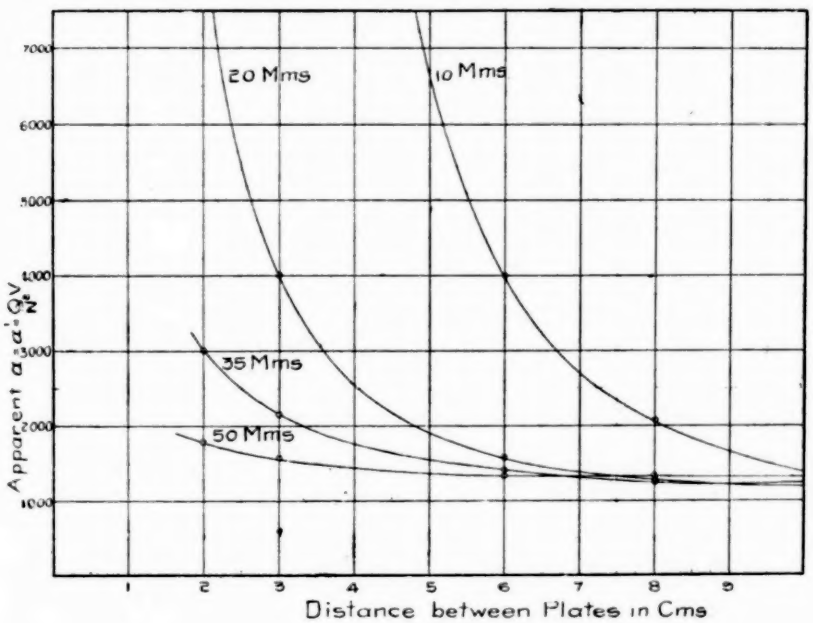


Fig. 4.

value of N_D compared with N the less the effect of diffusion. Thus at 50 mm. pressure and the plates 8 cm. apart, diffusion plays very little part, but at 10 mm. pressure for the plates 2 cm. apart it accounts for nearly the entire loss of ions.

For pressures below 10 mm. pressure and with vessels of ordinary dimensions as in discharge tubes, for instance, practically all of the ions disappear by diffusion to the sides of the vessel.

The Coefficient of Diffusion at Low Pressures.

For 20 mm. and 10 mm. pressures and for the small distances between the plates diffusion has been seen to play the important

part. Proceeding in a manner analogous to the method adopted for getting a separate from the diffusion error,it is possible to get a fairly good value for the coefficient of diffusion D at these pressures. Thus it is evident that the coefficient of diffusion, which is given by the equation

$$D = \tfrac{1}{3} \frac{Q}{N} a^2,$$

when there is no recombination, is a constant at any pressure and should not depend upon the distance between the plates. If, using the experimental values for Q and N, D apparently increases with the distances between the plates, it means that, due to recombination, the experimental N is smaller than it would be for diffusion loss only. The relative loss due to recombination should increase with the distances between the plates. Fig. 5, plotted from the values in the tables VII. and VIII. shows the increase of this apparent D with the distances between the plates.

TABLE VII.

Pressure, 20 *mm.*

Distance Between Plates.	Q.	N.	$\frac{1}{3}\frac{Q}{N}a^2 = D'$.
8 cm.	6.4	4.75	7.1
6	4.7	3.14	4.4
3	3.0	1.14	1.9
2	2.3	.52	1.4

TABLE VIII.

Pressure, 10 *mm.*

Distance Between Plates.	Q.	N.	$\frac{1}{3}\frac{Q}{N}a^2 = D'$.
8 cm.	3.4	2.7	6.8
6	2.38	1.4	5.0
3	1.6	.38	3.2
2	1.21	.143	2.8

As in the curves for a, where the D' becomes constant with the distance between the plates is taken as the coefficient of absorption at that pressure. In this manner D was found to have the value 11. at 20 mm. pressure and 2.6 at 10 mm.

Reducing these values back to atmospheric pressure on the assumption that D varies inversely with the pressure, we get the values .030 and .034. Townsend found (*loc. cit.*) that D varied inversely with the pressure down to 200 mm. pressure and his value at atmospheric pressure was .035. The two values given above

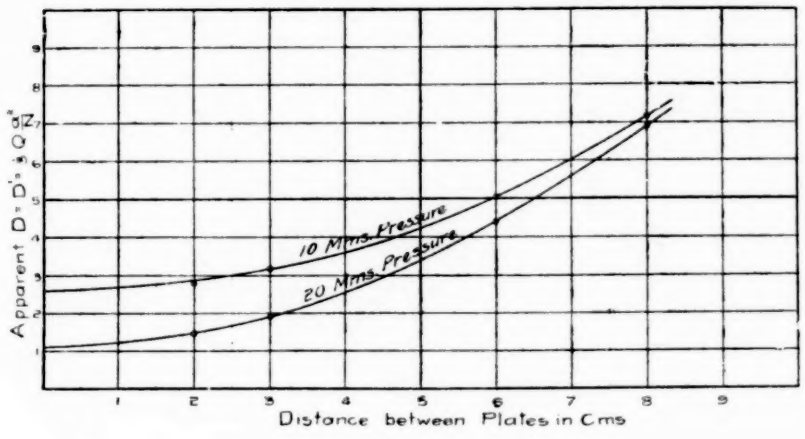


Fig. 5.

thus indicate that D continues to vary inversely with the pressure at least down to 10 mm.

As, by theory, D varies directly with the specific velocities of the ions this would indicate that the velocities too are a linear function of the pressure down to 10 mm. J. Stark[1] gives some results showing this, but Langevin[2] has found that the velocities increase with decrease of pressure faster than a linear law would allow.

Variation of α with the Pressure.

The complete results for the variation of α with the pressure are given in Table IX. and plotted in the form of a curve in Fig. 6. The values of α here given are the mean of several determinations. The values of N given in the table are not in every case the ones from which the corresponding α was obtained, as some of the determinations were made before the radium had reached a steady state.

[1] J. Stark, Die Electricität in Gasen, p 261.

[2] M. Langevin, Thèse, p. 140.

TABLE IX.

$V = 3{,}435$ c.c.

Pressure.	Q.	N.	$a = \frac{Q}{N^2} V$.
760 mm.	106	10.5	3,300
450	94	11	2,650
250	67	10.6	2,070
150	42.2	9.1	1,750
100	28	7.9	1,550
50	14.2	6.1	1,310
35	10	5.2	1,250
20	5.6	4.1	1,150
10	3.0	3.2	1,000

At atmospheric pressure the value of a has been previously well fixed. Townsend obtained 3,400, McClung 3,384 and Langevin 3,200 as the absolute value of a. The value 3,300 given here is seen to agree very closely with these results.

As to the variation of a with the pressure the curve in Fig. 6

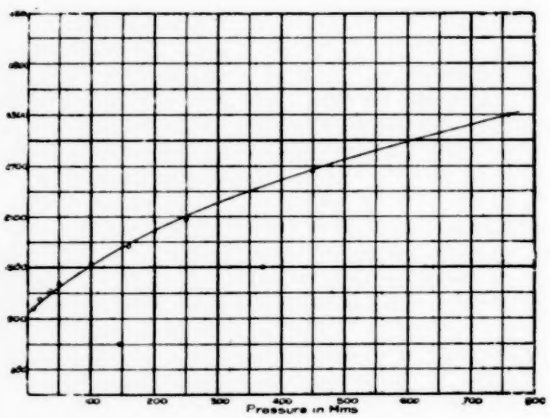


Fig. 6.

Coefficient of Recombination a.

shows that a decreases to a value 1,000 at 10 mm. pressure, the rate of decrease constantly increasing as the pressure falls.

The curve falls below that of McClung's who found a independent of the pressure down to about 100 mm. It goes above that of Langevin who found a somewhat more rapid decrease. Lan-

gevin does not give the absolute value of α obtained at pressures below atmospheric, but does give the data for 375 and 152 mm. for which the values 1,500 and 600 have been calculated. These values are indicated in Fig. 6 by two × marks.

It is difficult to compare the present results as to accuracy with those of Langevin as his experimental work was based on a rather complicated theory. In a general way Langevin's results are confirmed as there is a decrease of α with the pressure.

With McClung's results however a comparison can be made as both methods were direct ones. Langevin makes the following legitimate criticisms of the experimental conditions under which McClung worked.

The intensity of the Röntgen rays vary considerably in very short intervals of time, thus making it necessary to take the average of very many widely divergent readings. Also the ionization is very small even at high pressures, rendering accurate determinations of the charges difficult.

An error is introduced by not knowing the exact distribution of the ionization. Also the effect produced by the secondary radiations emitted by the metal plates under the action of the Röntgen rays is considerable and continues after the Crookes tube has been stopped.

Loss of ions by diffusion to the electrodes also brings in a probable error.

This last source of error might explain to some extent McClung's large values of α for low pressures, as he had a very large surface exposed and the electrodes were close together. This diffusion effect has been shown to come in very rapidly as the distance between the plates and the pressure decrease.

Practically all of these objections except that due to uneven distribution have been met in the present work. The intensity of the radium radiations was very constant and the ionization large and thus the readings could be repeated within a few per cent. The secondary radiations could introduce no error on account of the method of determining N. The diffusion effect was corrected for in those cases where it was appreciable. As for the distribution error, it has been shown that with a somewhat large divergence

from a uniform distribution the correction for α is very small, therefore any error in the determination of the distribution could not effect the value of α very much.

In conclusion I wish to acknowledge my indebtedness to Dr. Bergen Davis at whose suggestion this investigation was taken up and with whose advise it was carried on.

PHŒNIX PHYSICAL LABORATORY,
COLUMBIA UNIVERSITY, May, 1905.

NOTE ON THE IRREVERSIBILITY OF THE HEUSLER ALLOYS.

By Bruce V. Hill.

BEING attracted by the accounts of Heusler's magnetic alloys of copper, manganese and aluminium, the author cast several of these alloys which showed some properties so interesting that it seemed worth while to give a brief preliminary description of them.

It may not be out of place to say something of the history of the discovery of these alloys and of the theory which applies to such cases. It is common to think of iron, cobalt, nickel and bismuth as the only substances having a measurable magnetic susceptibility. As a matter of fact, however, a large number of the elements show either para- or dia-magnetic properties. Stephen Meyer[1] found that some of the rare metals are strongly para-magnetic. The metals themselves were very difficult to obtain in a pure state and in quantities sufficient for the purposes of testing. When the salts of some of them were examined they exhibited even a greater susceptibility than those of iron. Defining the *atomic susceptibility* as the mean susceptibility of a space containing one gram atom of the substance in 1,000 cubic centimeters, Meyer gives the atomic susceptibilities of a number of the elements in the following order: Ho, Er, Gd, Mn, Fe, Sa, Co, Yt, Nd, Ni, Pr.[2] He says that if then, the susceptibility of metallic erbium were as much greater than that of iron as the susceptibility of the oxide is greater than that of the oxide of iron, dynamo-machinery and other apparatus could be built having only one sixth the weight of those of iron for the same output, since the density of erbium is only two thirds that of iron. This premise is not necessarily true, however, for manganese also stands before iron in the above list and it is not magnetic in the ordinary metallic form, though electrolytic manga-

[1] Stephen Meyer, Ann. d. Phys , 68, p. 325, 1899.
[2] Stephen Meyer, Ann. d. Phys., 69, p. 236, 1899.

nese was found to be slightly magnetic. It may be possible to account for the different magnetic properties of these metals in their salts and in the metallic condition in a simple manner however.

So far as iron and nickel, at least, are concerned there seems to be abundant reason to believe that they can exist in several allotropic forms. This is the "allotropic theory" of Osmond. The facts supporting it can not be enumerated here; a number are given with references in my paper mentioned under 8. In iron there are supposed to be three such modifications: the alpha form, soft, magnetic and stable below 700° to 750° C.; the beta form, hard, unmagnetic and stable between 750° and 860° C.; and the gamma form, soft, unmagnetic and stable above 860°C. In passing one of these transformation points heat is evolved as is seen in the recalescence of steel. In nickel there have been found but two allotropic forms: the alpha form, magnetic and stable below 340° to 360° C., and the beta form, unmagnetic and stable above that temperature.

This temperature of transformation may be depressed by alloying a second metal with the iron or nickel. This depression is proportional to the amount of the alloy as is seen in the case of the alloys of aluminium and iron[1] or of copper and nickel.[2] In this the alloys behave as other solid solutions.[3] This depression of the transformation point is analogous to the depression of the freezing point of a solvent by the dissolved substance.

The allotropic theory explains the phenomena of the tempering of steel as the older theory that the hardening was due to strains set up in the metal by the sudden cooling never did. The compound steels furnish much of interest in this connection, especially the self-hardening and irreversible nickel-steels.[4] The irreversible steels[5] (so-called from the fact that if, starting in the magnetic condition, they be heated to dull redness they become unmagnetic, but on cooling do not regain their magnetic properties until a tempera-

[1] Lownds and Richardson, Phil. Mag. (6), 1, p. 601, 1901.

[2] B. V. Hill, Verh. d. Deut. Phys. Gesell. Jahrgang, IV., Nr. 9, 1902.

[3] Bruni, Feste Loesungen, German edition, Stuttgart, 1901.

[4] For a full account of these steels see L. Dumas, Recherches sur les aciers au nickel a haute teneures, Paris, 1902.

[5] Ewing, Magnetization in Iron and Other Metals, p. 185, 1900.

ture is reached much lower than that at which they originally became unmagnetic; if the temperature be again raised the material remains magnetic till the original point of transformation is attained) make it possible to compare the properties of these alloys in the magnetic and the unmagnetic conditions at the same temperature. This has been done by a number of experimenters.[1] It appears also that while one allotropic form, the magnetic, is that ordinarily existing at room temperature, it is possible to render the other, the unmagnetic, stable at that temperature.

It has been seen that the salts of manganese and chromium are in the same class, so far as the magnetic properties are concerned, with those of iron, cobalt and nickel. Might it not then be conjectured that manganese and chromium also have several allotropic forms or phases, of which it is the unmagnetic with which we are acquainted, but that, if the right alloy could be found, the magnetic form might be had.

Heusler's[2] discovery of such an alloy in the case of manganese seems to have been accidental, however. When working with an alloy prepared for another purpose and with a tool of whose magnetization he was not aware, he found that the alloy was attracted by the tool. He then sought in a systematic way to produce magnetic alloys of manganese. He found that certain of the trivalent elements have a particularly strong effect upon manganese even at comparatively low temperatures. If one part of powdered antimony and four parts of powdered manganese be heated together to 500° or 600° C., which is still far below the melting point of manganese, the mixture becomes noticeably magnetic. I tried a similar experiment with chromium, but without success.

Starting with the commercial alloy of 70 per cent. copper and 30 per cent. manganese, Heusler made several series of alloys using as a third metal tin, antimony, arsenic, aluminium, etc., of all of

[1] Ch. Ed. Guillaume, Comt. Rend., 124, p. 176; 1515, 1897; 125, p. 235, 1897. E. Dumont, C. R., 126, p. 741, 1898 L. Dumas, C. R., 130, p. 1311, 1900. F. Osmond, C. R., 118, p. 532, 1894; 128, p. 304, 1899. H. Tomlinson, Proc. Roy. Soc., 56, p. 103, 1894. J. Hopkinson, Proc. Roy. Soc., 47, p. 23; 138, 189; 48, p. 1; 442, 1890, 50, p. 121, 1891. B. V. Hill, Verh. d. Deut. Phys. Gesell., III., Nr. 10, 1901.

[2] Schriften d. Gesell. z. Befoerderung d. gessammt. Naturwiss. zu Marburg, Bd. 13. 5te. Abtheilung, Marburg, 1904. (Verh. d. Deut. Phys. Gesell., 5, 12, p. 220, 1903.)

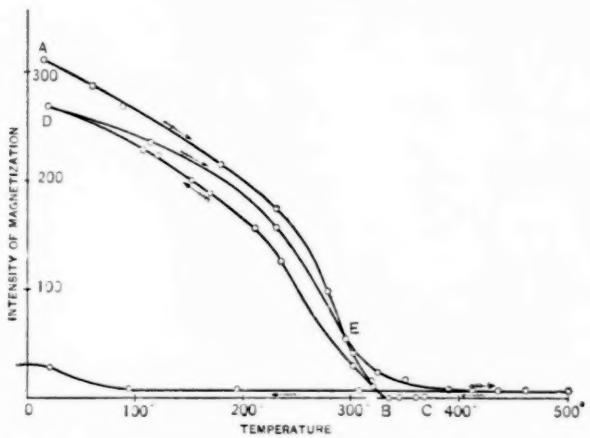
these the last gave the best results. In the alloys of copper, manganese and aluminium the susceptibility increased until the masses of the manganese and aluminium were in the ratios of their atomic weights, and then grew less again as the amount of aluminium was increased. The magnetic properties of these alloys were investigated by W. Starck and E. Haupt,[1] the magnetic expansion by L. Austin[2] and the effects of prolonged heating and cooling by E. Gumlich.[3] Continued heating at a temperature of 110° C. increased the susceptibility, while annealing at 500° C. left the material practically unmagnetic. Cooling in liquid air had no effect. None of these authors mention a temperature over 580° C.

Of the alloys made by the author one contained about 60 per cent. copper, 25 per cent. manganese and 15 per cent. aluminium, the other a higher per cent. of aluminium. They were made by melting the aluminium with the 30 per cent. manganese-copper alloy, and were cast in the form of rods 0.45 cm. in diameter and 12 cm. long. The metal was very hard and brittle. The magnetic testing was done by the ballistic method. About a porcelain tube 70 cm. long and 2.5 cm. in diameter was wound a nickel heating coil. This was put into a glass tube 5 cm. in diameter and packed about with kaolin. The glass tube fitted closely into a brass water jacket having a space 0.5 cm. between the walls. The magnetizing coil was wound upon this jacket and could thus be kept at a constant temperature even when the interior of the apparatus was very hot. The secondary coil was would upon a smaller porcelain tube which was placed within the heating tube. The specimen, together with one junction of a thermo-element, was within this second porcelain tube and packed with asbestos to prevent heat losses. Corrections were of course made for the demagnetizing effect of the ends of the rod and for the area between the rod and the secondary coil. It may be remarked here that an apparatus of this kind is intended to measure changes of induction with temperature and does not profess to give absolutely accurate values of the induction itself though the error in this respect is not large. M. L. Dumas[4] deter-

[1] Verh. d. Deut. Phys. Gesell., 5, 12, p. 224, 1903.
[2] Verh. d. Deut. Phys. Gesell., 6, 14, p. 211, 1904.
[3] Ann. d. Phys., 16, p. 535, 1905.
[4] See 6.

mined the transformation points of his nickel-iron alloys by merely observing the temperature at which a compass needle near the specimen ceased to be affected by it.

The first rod tested was that of higher percentage of aluminium. It showed an induction of $B = 5750$ c.g.s. in a field on $H = 87$ c.g.s. with a comparatively large area of the hysteresis curve. On being heated it behaved as those of the other observers had done (up to 500° C.), yet it may be of interest to give the curves showing the relation of the intensity of magnetization, I, to the temperature as the metal is heated and cooled, since no such have been published. The field was kept constant and equal to 85 c.g.s. A relatively strong field was used to avoid the occurrence of maxiam



as in the curves for iron and nickel in weak fields.[1] The general form of the curves here is similar to that for the other magnetic metals and their alloys. Beginning at A with an intensity of $I = 311$ c.g.s. the curve falls till it reaches the axis at B. The temperature was carried 33° higher bringing the curve to C. On cooling the line leaves the axis again at B but does not return to A but to a lower value at D, $I = 267$. With the second heating the curve lies above that for the first cooling and does not come to the axis at B but crosses the former heating curve at E and when

[1] Ewing, Mag. in Iron, Figs. 79, 85, 8 , 1900.

heated to 500° C. retains a slight susceptibility, I being equal to 7. After falling the second time to 17° C. the intensity was only 27, that is the susceptibility was only 0.32. A reading was taken with the specimen in liquid air, giving $I = 36$. This remained unchanged when room temperature was again reached.

It now occurred to me that, from the general behavior of the alloy, it might be expected that the magnetic qualities of the metal would be restored if it were taken to a temperature considerably higher than those previously attained. The specimen was, therefore, heated, in the flame, to dull redness (650° to 700° C.) and allowed to cool slowly. When tested in the same way as before the intensity was $I = 90$ in the field of $H = 85$. Reheating to the same temperature and cooling rapidly did not alter this value of I. The rod was then heated to a bright red (850° C.?) and when tested at room temperature the intensity was $I = 155$. After cooling in liquid air $I = 162$.

We have here then a new type of irreversible alloy, but in a sense opposite to the irreversible nickel-steels. Compare this with one of the specimens used in the study of the specific heats of some of these nickel-steels.[1] The alloy contained twenty-four per cent. of nickel. Let B_H represent the induction B for the field H at the temperature of the room. The temperatures given are those to which the metal was cooled, or heated, immediately before the induction was determined. For the sake of the comparison $B = H + 4\pi I$ is used instead of the values of I given in the preceding paragraphs. Tabulating the results for the two alloys:

Nickel-steel,				
When received,	0°	− 20°	− 79°	− 190°
Unmagnetic,	$1{,}200_{136}$	$3{,}700_{135}$	$9{,}200_{135}$	$10{,}200_{136}$
Heusler alloy,				
When cast,	368°	500°	650°?	850°?
$4{,}020_{85}$	$3{,}440_{85}$	427_{85}	$1{,}220_{85}$	$2{,}030_{85}$

In the first case the alloy, after cooling from the temperature of forging, was unmagnetic, but its permeability was found to be

[1] Verh. d. Deut. Phys. Gesell., III., Nr. 10, 1901.

greater the lower the temperature to which it had been cooled. Heating to dull redness left it again unmagnetic. In the other alloy the metal was magnetic when first cast. Cooling to very low temperatures had only a very inconsiderable effect. If heated and allowed to cool to room temperature, the permeability varied with the temperature to which it had been carried, showing a minimum in the neighborhood of 500° C. One might venture a further comparison, as the merest conjecture. In the nickel-steel here spoken of, the magnetic state, when cooled from the temperature of the furnace, is that of iron at high temperatures. Dr. D. K. Morris[1] found also that iron regains its susceptibility to a slight degree between 800° and 1000° C. It is then possible that manganese would be found to be magnetic at a sufficiently high temperature.

Returning to the description of the alloys in hand, the second rod tested was that of lower per cent. of aluminium. It had an induction of $B = 11{,}800$ in a field of $H = 75$, with very little hysteresis. Its behavior was in general like that of the first one, but, when heated to 950° C. and cooled to room temperature, it remained unmagnetic. There was a great difference in the density of this alloy in the magnetic and in the unmagnetic conditions. When first cast the density was 6.61. After annealing at 950° C. it was but 5.80. Here again the nickel-steels show an opposite change. One of the alloys of nickel and iron referred to above had a density of 7.91 in the magnetic state, 8.09 in the unmagnetic, a change of about 2 per cent. This agrees with the results of H. Tomlinson[2] and J. Hopkinson.[3]

Stephan Meyer found that, when a diminution of volume occurs in a compound, the para-magnetic character of the substance is increased and *vice versa*. In the case of alloys we are, in general, dealing with solid solutions and not with compounds of the constituents, so that the applicability of this statement to alloys would have to be tested by experiment. There is possibly more reason to suspect chemical combination between the manganese and the aluminium in the Heusler alloy than between the iron and the

[1] Phil. Mag., 44, p. 213, 1897.
[2] See footnote No. 8.
[3] See footnote No. 8.

nickel in the nickel-steel, since the permeability of the former is greatest when the percentages of the two metals are proportional to their atomic weights. These alloys are also those which are most dense in the magnetic condition.

ON THE WAVE-LENGTH OF THE RADIATION GIVEN OFF IN AN ALTERNATING CONDENSER FIELD.

By Fernando Sanford.

IN a previous paper by the present writer[1] some measurements were given of the wave-length of the radiation from aluminum, silver, copper and zinc when these metals are joined to the cathode of the secondary of an induction coil or are placed in the field of an air condenser the plates of which are joined to the terminals of an induction coil. Since that time, similar measurements have been made for other metals and for some non-metallic substances. These measurements have been made by means of the grating used in the former experiments, viz., a celluloid replica of a Rowland plane grating having 14,438 lines to the inch.

In all cases here mentioned the radiating substance, the grating and the photographic plate were enclosed in a dark box which was placed between the glass plates upon which the tinfoil condenser sheets were pasted. The radiation was allowed to pass through a slit in a vulcanite screen and through the grating and fall upon the photographic plate. On account of the feeble intensity of the radiation, the slit and grating were placed only from four to five centimeters from the plate, and the distance of the first spectrum from the central image of the slit was only from eight to ten millimeters. Naturally this distance could not be measured with a high percentage of accuracy, but the plates were all re-measured by Miss Shirley Hyatt, a student in the laboratory, and the two measurements seldom differed by more than two or three per cent.

In general, the radiation from each metal seemed very homogeneous. Only in the case of platinum was the spectrum appreciably wider than the central image of the slit. The metals tested and the wave-lengths determined for their radiations are given below.

[1] Physical Review, XVIII., 366, May, 1904.

Metal.	λμμ.	Metal.	λμμ
Cadmium,	335	Nickel,	360 (?)
Tin,	340	Iron,	369
Bismuth,	346	Copper,	372
Antimony,	351	Zinc,	380
Magnesium,	348 (?)	Manganese,	380
Aluminum,	350	Lead,	380
Silver,	350	Platinum,	380–400
Gold,	350	Tinfoil,	329

The measurements made on magnesium and nickel were less satisfactory than the others, the photographs of the spectra being less clear. In platinum the width of the diffracted image indicated that the radiation was less homogeneous than in the case of the other metals. Apparently some of this radiation extended to the extreme violet of the visible spectrum. Carbon disulphide was, however, apparently quite opaque to the platinum radiation, indicating that the greater part of the radiation was of wave-length not much greater than 380.

A number of crystals of various salts were tried as radiators by sticking a crystal with glue on a piece of cardboard which was placed upright in the dark box in front of the slit. It was found that in this position the cardboard gave off no appreciable radiation. Of the crystals tried, all gave off the radiation except sodium chloride and nickel sulphate. These gave no appreciable radiation after exposures of from 17 to 25 minutes while from 5 to 10 minutes was generally long enough to give a plain image of the slit. It is interesting to notice in this connection that sodium chloride is one of the salts found by Schmidt[1] not to be actinoelectric. Nickel sulphate is not given in his list.

The following substances gave off radiation of wave-length approximately as given below:

Substance.	λμμ.	Substance.	λμμ.
Silver nitrate,	334	Potassic bromide,	370
Ferrous sulphate,	weak (?)	Potassic ferrocyanide,	334
Copper sulphate,	363	Cinnabar,	359
Lead nitrate,	348	Sulphur,	(?)

[1] G. C. Schmidt, Wied. Ann., 64, 708–724.

In the case of radiation from crystals even more than from metals it was found that sharp points or edges were the principal sources of the radiation. Photographs of crystals made by means of their own radiations and by the use of a quartz lens showed only a number of point or line sources of the radiation.

STANFORD UNIVERSITY,
June, 1905.

TOTAL REFLECTION.

BY ELMER E. HALL.

IN a former article[1] the writer has discussed the theory of penetration during total reflection, when reflection takes place from a thin film contained between two denser media. The discussion was based upon the electromagnetic theory of light and particular application was made to the case of a film between the hypotenuse surfaces of two right-angled glass prisms placed almost in contact. If the electric forces be regarded as the light vectors, the ratio of the intensities in the two glass prisms can be obtained from the ratio of the electric forces within the prisms. Neglecting losses due to absorption, and representing the component of the electric force perpendicular to the plane of incidence in the first and second prisms by E_s and D_s respectively, it has been shown that[2]

$$\frac{D_s^2}{E_s^2} = \frac{4\cos^2\varphi\left(\frac{\sin^2\varphi}{n_1^2} - 1\right)}{(1-n_1^2)(n_2^2-1)\sinh^2 u + (\cos\varphi + n_1 n_2\cos\theta)^2\left(\frac{\sin^2\varphi}{n_1^2} - 1\right)}. \quad (1)$$

Here φ is the angle of incidence, within the first prism, on the boundary of the film; θ is the angle of refraction within the second prism; n_1, n_2 are the indices of refraction from the first prism into the film and from the film into the second prism respectively;

$$u = \frac{2\pi d}{n_1\lambda}\sqrt{\sin^2\varphi - n_1^2},$$

d being the thickness of the film and λ the wave-length of light within the film. The above equation expressing the ratio of the squares of the electric forces gives also the ratio of the squares of the amplitudes of the incident and emergent light. This will be the ratio of the intensities only when the two prisms have the same

[1] PHYSICAL REVIEW, Vol. XV., p. 91.

[2] Loc. cit., p. 97.

index of refraction. If n_1 is not numerically equal to $1/n_2$ the relative intensities will not be measured by the squares of the amplitudes merely, but the rates at which the energy is propagated by the corresponding waves in the two media must be taken into account. In this case, if the two media enclosing the film be different, the ratio of the intensities will be

$$\frac{D_s^2}{E_s^2}\tan\varphi\cot\theta = \frac{4\cos^2\varphi\left(\frac{\sin^2\varphi}{n_1^2}-1\right)\tan\varphi\cot\theta}{(1-n_1^2)(n_2^2-1)\sinh^2 u+(\cos\varphi+n_1n_2\cos\theta)^2\left(\frac{\sin^2\varphi}{n_1^2}-1\right)}. \quad (2)$$

If now the beam of light be reversed, φ and θ will interchange and $1/n_1$ replace n_2 and *vice versa*. Inserting these changes in (2) the ratio of the intensities within the two prisms becomes

$$\frac{D_s'^2}{E_s'^2}\tan\theta\cot\varphi = \frac{4n_1^2n_2^2\cos\theta\,(n_2^2\sin^2\theta-1)\tan\theta\cot\varphi}{(1-n_1^2)(n^2{}_2-1)\sinh^2 u'+(n_1n_2\cos\theta+\cos\varphi)^2(n_2^2\sin^2\theta-1)}. \quad (3)$$

Since

$$n_2^2\sin^2\theta=\frac{\sin^2\varphi}{n_1^2}$$

the second members of (2) and (3) may be reduced to an identity. It therefore follows that the penetration for any such system is the same whether the light passes from the denser to the rarer of the two media enclosing the film or in the reverse direction. Similar treatment yields the same conclusion for light polarized perpendicular to the plane of incidence. In the former discussion of the theory the factor $\tan\varphi\cot\theta$ was omitted and hence erroneous conclusions drawn.[1]

To test the conclusion under better experimental conditions than in the former measurements, a flint-glass prism, whose index of retraction for sodium light was 1.7597, was used with a crown-glass

[1] I am indebted to Professor Voigt for calling my attention to the error, by letter and also in Nach. d. K. Gesellschaft d Wissenchaften zu Gött. 1903 Heft 3 S. 1 where a different treatment yields the same conclusion.

prism of refractive index 1.5164. The hypotenuse surface of the crown-glass prism was slightly convex. The prism angle for the flint-glass prism, 27° 54′, was such that when the two prisms were used in combination no deviation was obtained with sodium light. The source was a narrow yellow band from the spectrum of a Nernst lamp filament, the current being maintained practically constant by means of an electrodynamometer and an adjustable resistance. The following table gives the average of a number of closely agreeing readings. The light was polarized perpendicular to the plane of incidence; d is the depth of penetration in microns; i_1 and i_3, for the same line, are so related that i_3 is the angle of emergence from the flint-glass when i_1 is the angle of incidence on the crown-glass prism.

Crown-glass, Air, Flint-glass.				Flint-glass, Air, Crown-glass.			
i_1	θ	d	d/λ	i_3	θ	d	d/λ
5°	40° 42′ 20″	1.98	3.36	12° 30′ 00″	34° 58′ 45″	1.85	3.15
3	43 1 30	1.32	2.24	14 22 30	36 0 45	1.28	2.18
0	45 0 0	1.06	1.80	17 8 00	40 2 20	0.99	1.69

The differences in the values for the two cases are greater than the experimental errors. Repeated sets of measurements, taken with different intensities, all gave the same order of differences for the two cases. Light polarized parallel to the plane of incidence also gave the same result, the penetration is apparently greater when the light is incident on the crown-glass. The intensity observed, however, is not the intensity in the second prism but the intensity in air after the light has left the second prism, and futhermore it is the intensity in air before incidence on the first prism and not the intensity of the light incident on the film that is constant. The Fresnel coefficients for transmission from air to the first prism and from the second prism to air must be taken into account. The intensity of the light transmitted by the system for any given angle of incidence is not the same when the light is made to travel the same path, but in the reverse direction, unless the amount of light getting through the film varies directly as the intensity of the light incident on the film. This, however, is not the case, the penetration is a complicated function of the intensity.[1] For the condition under

[1] PHYSICAL REVIEW, Vol. XV, p. 100.

which the measurements were taken the results for the two directions therefore will not be identical. The penetration will be the greater for that direction in which the light incident on the film has the greater intensity. Using Fresnel's coefficient for light polarized perpendicular to the plane of incidence, the intensity of the light incident on the film when crown-glass is the first medium, for $i_1 = 5°$, is to the intensity of the light incident on the film when the flint-glass is the first medium, for the corresponding angle of incidence $i_3 = 12° 30'$, in the ratio of 959 to 929. Absorption has not been taken into account. Since the flint-glass has a distinct yellow tinge the losses due to absorption would increase rather than diminish the ratio of intensities as given. The greater value for the penetration obtained in the above table, when light was incident on the crown-glass, is thus accounted for, and it may be concluded that the penetration is independent of the order in which the light passes through the prisms.

UNIVERSITY OF CALIFORNIA,
BERKELEY, July 5, 1905.

NEW BOOKS.

Maxwell's Theory and Wireless Telegraphy. Part I. by H. POINCARÉ. Part II. by FREDERICK K. VREELAND. 8vo, pp. xi + 255. New York, McGraw Publishing Co., 1904.

The first part of this book is a translation, by Mr. Vreeland, of Poincaré's "la Theorie de Maxwell et les Oscillations Hertziennes" which appeared in 1903 as one of the excellent "Scientia" series published by Naud. Beginning with an explanation of the essential features of the Maxwell theory, this portion of the book contains an account of electrical oscillations and electric waves considered from the scientific standpoint solely. The whole is written in the clear and pleasing style which characterizes all of Poincaré's writings.

The reader who knows Poincaré as a mathematician and mathematical physicist will perhaps be surprised at the complete absence of mathematical methods in the discussion. It is surprising also that so difficult a subject as Maxwell's theory can be so successfully presented without mathematical aid. M. Poincaré has given us another illustration of the significant fact that the greatest theoretical physicists have always been able to write clearly in ordinary language and in concrete terms. Every theoretical physicist must make frequent use of analytical methods: in the solution of problems as labor saving machinery, and in the statement of results as a convenient and useful shorthand. But the body is more than the raiment: a physical theory is more than the analytical expression with which we clothe it; and it is only the scientist of lesser caliber who is unable to separate the two.

M. Poincaré has made extensive use of mechanical analogies in helping to clear up difficult points. The translation here has a distinct advantage over the original in the fact that numerous illustrative diagrams have been inserted. The illustrations in the original were quite inadequate.

Wide differences of opinion exist regarding the benefit to be obtained from mechanical models in the study of electrical subjects. When the models are of such a character that the student thinks of them as actually representing the underlying mechanism of electrical phenomena, and proceeds to fill up space with an ether made up of cog wheels and spiral springs, I am inclined to agree with those who think that more harm than good is done by their employment. But M. Poincaré's models, and those introduced later by Mr. Vreeland, are not of that kind. Take for example the hydraulic analogy and the pendulum analogy, which are so effectively used in explaining the fundamental laws of electrical oscil-

lations; or the mechanical models used by Vreeland later on in explaining the different methods of coupling the antenna to the primary vibrating system. There is not one reader in a thousand whose understanding of the subject will not be made clearer by the consideration of such mechanical parallels. Yet in all cases the model is so different from the reality that there is no danger of thinking of it as giving a picture of the actual processes. Such analogies as this are the parables of science; they convey as much in a sentence or a diagram as could be explained in several pages of exact statement.

After M. Poincaré's clear and adequate treatment of the scientific aspects of electric waves, we have in the second part of the book, by Mr. Vreeland, an almost equally satisfactory presentation of the principles of wireless telegraphy. The plan and scope of this part of the book are explained by a few sentences from the preface: "In Part II. it has been my purpose to take up the thread where M. Poincaré dropped it, carrying the line of thought into the practical field of wireless telegraphy, and applying the principles laid down in Part I. to the various problems involved; to describe certain typical systems, to show why some have failed while others succeeded, and to explain their mode of operation in the light of Maxwell's ideas. This is not intended as a treatise on wireless telegraphy — no attempt is made to describe the myriad forms of apparatus nor to settle questions of priority and history. * * * The object is rather to deal with principles and to trace the development of the art in its essential features."

Accounts of practical wireless telegraphy have been so often either highly technical or absurdly superficial that a treatment of the subject according to the plan outlined above is very welcome. The book contains no elaborate photographs of apparatus such as are often presented in our technical press as adequate illustrations of new systems: usually a confused jungle, in which induction coils and high tension transformers rise at intervals from an undergrowth of choke coils, relays, and sounders; the connections left to the imagination, and all really essential features concealed in camera-proof boxes. On the contrary the more important methods and systems are explained by the aid of simple diagrams of connections, and by reference to well chosen mechanical analogies. The essentials are there, and are explained in a sound and concrete way. Unessential details, which would only confuse the reader by their added complexity, are either left out of consideration altogether or are referred to only briefly. The author does not forget that he has set out to explain the *principles* of wireless telegraphy. In an art so new as this the forms of apparatus may change before a book describing them is off the press; but the underlying principles of telegraphy by electric waves will remain unchanged.

Mr. Vreeland's practical experience in wireless telegraph work gives him an advantage in preparing such a book that is often in evidence. What impresses me perhaps most as one of the results of practical experience is the manner in which the many difficulties which a careful reader would encounter are met and explained almost before they arise. It is evident in many cases that the author has met these difficulties himself in concrete form. Mr. Vreeland has wisely avoided discussion of the cases of disputed priority which are unfortunately so numerous. His references to the work of different inventors are fair and remarkably free from the bias that would be natural in one who has been intimately connected with one particular system out of the several competing ones.

The plan followed in Part II. is indicated by the following titles of chapters: General Principles; Telegraphy by Hertzian Waves; the Grounded Oscillator; Propagation of Grounded Waves; Receiving Apparatus; Selective Signaling. In the opinion of the reviewer the book is the most satisfactory that has yet appeared upon the subject of wireless telegraphy.

ERNEST MERRITT.